# Communication, Feedbacks and Repeated Moral Hazard with Short-lived Buyers*

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**Abstract.** *We show that experience good sellers facing myopic buyers can solve the inherent moral hazard problem by communicating their observation of quality before trade, provided that communication is part of their public track record. Such cheap-talk communication, if trusted, allows market prices to reflect the actual value created, thus providing an immediate reward for the seller's effort which complements the conventional, reputational incentives. Pre-trade communication achieves maximal efficiency when truthful and the full efficiency as the noise in the seller's observation vanishes. We fully characterize the conditions for communication to improve efficiency and the extent to which it does so.* (*JEL* Codes: C73, D83, L14)

*Keywords:* cheap talk, moral hazard, reputation mechanism, trust.

# 1 Introduction

Solving moral hazard amounts to finding a way to reward the agent for exerting the socially efficient effort. In long-run market environments, forward-looking sellers may be incentivized by backloaded compensation schemes that price their goods in line with the quality they delivered in the past, exploiting *ex-post* monitoring of effort via delivered quality (Klein and Leffler, 1981; Shapiro, 1983).[1] However, such incentive schemes are impaired and fall short of achieving efficiency when buyers are short-lived and seller's effort is imperfectly monitored, no matter how patient the seller is.[2] We redress this deficiency

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[1]See MacLeod (2007) for a review of the related literature.

[2]Note that the Folk Theorem of Fudenberg, Levine, and Maskin (1994) does not apply when buyers do not engage in a long-run relationship with a particular seller (see Fudenberg, Kreps and Levine, 1990).

by establishing that the market pricing mechanism can restore efficiency in such situations if, in addition to the past quality supplied by a seller, her pre-trade communication on current quality is also properly reflected in pricing and in the trust level bestowed on the seller. To the best of our knowledge, this is the first result showing that cheap-talk communication can help solve moral hazard by rewarding hidden action without delay.

The importance of pre-trade communication in customer relationships is well documented in the marketing literature (Agnihotri et al., 2009; Palmatier et al., 2006). But, such soft communication by sellers on their good (e.g., online sellers describing the condition of their items and salesmen providing guidance to potential buyers) has largely been overlooked in the context of incentivizing sellers to exert the socially efficient effort.

The core insight of our paper stems from the simple observation that whatever sellers may know about the quality, provided that it is truthfully communicated, can help the market price their product closer to the actual quality. This would allow immediate compensation for the seller's effort in line with the social value created, which may complement the conventional, reputational incentives for effort. As such, truthful communication of a seller's observation of quality creates an immediate "efficiency rent" for the seller, easing the incentive constraint for her to exert the efficient effort.

However, truthful communication is at odds with the short-term incentive of claiming a better quality to get a higher price, thus it must be induced via long-term incentives. This is an extra condition, in addition to inducing efficient effort via reputational motives, that further constrains the ways in which long-term incentives may be devised if truthful communication were to be accommodated. Hence, it is unclear *a priori* whether seller's pre-trade communication may enhance efficiency and welfare.

We characterize precisely when and how much of such welfare improvement is possible. Broadly speaking, pre-trade communication enhances efficiency and welfare so long as the effort cost is neither too small nor too big, reaching full efficiency across the entire range of effort cost as the seller's observation of quality becomes fully accurate. Two key observations buttress such positive effects of communication. First, for the seller to preserve her efficiency rent, she must sustain buyers' trust in her communication. Second, truthful communication and efficient effort are strategic complements in the sense that the former is valuable only if the latter is intended in the future. Consequently, trustworthy behavior of the seller and the market's trust reinforce each other to uphold efficiency.

These findings lend implications on how the current feedback systems widely used in online platforms could be improved by suitably incorporating seller's pre-sale description of items. Online marketplaces have become a ubiquitous trading channel in less than two decades after Amazon and eBay opened in 1995, notwithstanding inherently weak trust between online traders due to their anonymity. Key innovations to tackle this trust issue have been reputation and feedback mechanisms that allow traders to leave evaluations on their counterparts for prospective future traders (Dellarocas, 2003).[3] Another prominent feature of online platforms is that prospective buyers rely on the soft information provided by the seller on attributes and quality of her products. Our findings underscore the importance of properly designing communication channels and rewarding faithful communication through reputation systems.[4]

Specifically, we analyze the effect of pre-trade communication by a seller who repeatedly produces an experience good of random binary quality subject to moral hazard, observes a noisy signal on the produced quality and may communicate about it by cheap talk before selling the good in a market of short-lived buyers. The seller's track record of past communication and delivered quality is assumed available for potential buyers, which typically is the case (or feasible) in online markets.

To determine the maximum extent to which the socially efficient effort can be sustained in equilibrium, we represent equilibrium value as a "self-generated" value in the sense developed by Abreu, Pearce and Stacchetti (1990) and Fudenberg and Levine (1994); then maximize the self-generated value subject to suitable incentive compatibility conditions. We characterize fully the solution of this problem which turns out to be a tractable linear program.

As a result we can clarify how the maximum achievable efficiency level varies depending on the severity of moral hazard, the cost of efficient effort, and how noisy the seller's signal is. Without communication, the maximum achievable efficiency falls short of the full efficiency level uniformly by an amount proportional to the cost of efficient effort, regardless of seller's patience. By contrast, with communication full efficiency is achieved when the noise on the seller's signal gets small, provided that she is patient enough. Therefore, pre-trade communication enhances welfare if the seller's signal is precise enough. It is also shown that the maximum efficiency with meaningful communication is achieved when the seller communicates truthfully.

As the seller's signal gets noisier, two countervailing forces contend: on the one hand, lying is more attractive as it is more likely to go undetected; on the other hand, it is less attractive because the price differential reflecting the seller's signal, which is the short-term gain from lying, dwindles. The former effect dominates except for an intermediate range of noise levels. Thus, as the seller's signal deteriorates from being perfect, the maximum efficiency achievable with truthful communication changes non-monotonically: it declines initially since truthful communication becomes harder to induce, then improves temporarily before declining again eventually.

[3]Cabral and Hortaçsu (2010) and Klein, Lambertz and Stahl (2016) find evidnce in eBay data that feedback system alleviates moral hazard.

[4]As an illustration, we note that two of the four options under eBay's rating system are related to communication: 'Item as described' and 'Communication'. Amazon has a codification of the state of used goods (from 'like new' to 'acceptable') and a dedicated section for "Customer questions & answers."

Turning to the cost of exerting the efficient effort, almost full efficiency is achievable without communication if the cost is very low, but not with truthful communication unless the signal is perfect. This is because imperfect monitoring of communication necessitates triggering punishment with a non-negligible probability to prevent lying, causing a non-negligible departure from the full efficiency. Hence, given any positive noise level, communication improves efficiency only if the effort cost is not too small, more precisely, when it is in an interior interval bounded away from zero.

In situations of severe moral hazard,[5] it can happen that this interval vanishes when the noise level in the seller's signal is in some intermediate range but reappears both when it is lower and higher (but not too high) owing to the non-monotonic effect of noise on efficiency mentioned above. Thus, an interesting, counter-intuitive implication ensues: efficiency may be enhanced by lowering the precision of the seller's observation of quality. We also provide a tight upper bound of the noise level below which communication can be beneficial, which is not very demanding generally.

We conduct our analysis presuming that the seller communicates about the observed signal, but she could also try to communicate about the effort level exerted. However, the latter communication is redundant and thus is ineffective insofar as the effort level is correctly anticipated in equilibrium. In contrast, communication on signal conveys new, *interim* information (not embedded in the equilibrium strategy) that can be used to compensate the hidden action on the spot and thereby, boost incentives. The underlying idea could be more general: falsifiable communication of *interim* information correlated with the chosen effort may enhance efficiency.

This observation points to a link with the standard insight from the relational contract literature (MacLeod and Malcomson, 1989; Baker, Gibbons and Murphy, 2002; Levin, 2003), namely, that incentives could be provided for the seller by delaying reward until after the quality is realized, through *ex-post* bonuses paid by a buyer voluntarily either directly or via higher future prices. Such a scheme however is not viable with short-lived buyers who would renege on any promised bonus. Our analysis suggests that pre-trade communication may allow substituting *ex-post* bonuses with instant bonuses based on the information revealed by the seller.

**Related literature**

The current paper contributes to the literature on trust and reputation. There are broadly two modelling approaches to seller reputation as a mechanism to incentivize their productive action. The first approach, elaborated by Klein and Leffler (1981) and Shapiro (1983) among others, rests on the idea that sellers are motivated to build a good track record in order to earn the trust of future buyers and thereby, a higher future income

[5] In the sense (to be made precise later) that the efficient effort is not easy to monitor via delivered quality, hence difficult to induce.

stream. If the seller's action is perfectly monitored *ex-post* via delivered quality, essentially a Folk Theorem obtains and full efficiency is sustained in equilibrium. We build upon this approach for the case that the seller's action is imperfectly monitored, and characterize the maximum efficiency achievable both with and without pre-trade communication.

The other one is the adverse selection-based approach pioneered by Kreps and Wilson (1982) and Milgrom and Roberts (1982). Adapted to our context, a patient seller obtains a payoff arbitrarily close to the efficient level by appearing as a "committed" type who is believed to always take the socially efficient action (cf. Fudenberg and Levine, 1992), but in equilibrium she mixes between efficient and inefficient actions and her reputation declines gradually. Positing instead an "inept" type from which a seller desires to be distinguished, Mailath and Samuelson (2001) show, *inter alia*, that full efficiency is sustainable if a seller's type is subject to change at any time so that the problem of dwindling reputational motive at very high reputation levels is precluded. In contrast, we show that full efficiency is achievable even without hidden seller types, by utilizing the seller's pre-trade announcement to align her income stream more closely with the value she creates.[6]

More broadly, we contribute to the literature on repeated games of imperfect public monitoring, shaped by such influential papers as Green and Porter (1984), Fudenberg, Levine and Maskin (1994), and Abreu, Pearce and Stacchetti (1986, 1990). Specifically, in the environments studied by Fudenberg and Levine (1994) where the power of the Folk Theorem is impaired due to myopia of short-run players, we show that efficiency can be improved via cheap-talk communication on endogenous private information.

We build upon the insight from Sobel (1986) and Morris (2001) that truthful communication may be motivated by the desire to preserve future credibility of communication. Recently, Best and Quigley (2017) study how the concern for future may enhance credibility of a sender who sequentially persuade short-lived receivers in the sense of Kamenica and Gentzkow (2011) but without commitment capability. Our mechanism differs critically because moral hazard is key to sustaining credibility. In our model communication doesn't steer the receiver into taking a better action for the sender—given the seller's effort choice, trade always occurs and the expected price is the same with and without communication—yet helps the seller self-discipline in providing quality. As such, without moral hazard at the production level, there is no room for communication. Thus, our contribution lies in uncovering the complementarity between two forms of moral hazard: the effort in delivering quality and the credibility in communication.

Moving to the related IO literature, Athey and Bagwell (2004) and Athey, Bagwell and Sanchirico (2004) show that *ex-ante* communication under adverse selection improves coordination in a collusive agreement. Awaya and Krishna (2016) show that *ex-post* com-

[6]Jullien and Park (2014) show that pre-trade communication permits prices to reflect quality better in pure adverse-selection settings as well.

munication improves private monitoring and helps sustain higher collusive prices. Our work differs in that we focus on *interim* communication under imperfect public monitoring without adverse selection. Also related is Rhodes and Wilson (2016) on false advertising as seller's announcement can be interpreted as advertising. They assume exogenous penalties for lying and exogenous quality, and focus on allocative inefficiency. Inderst and Ottaviani (2009) analyze firm's internal agency problem of incentivizing sellers to advise consumers through adequate compensation, under exogenous penalties for misselling. In comparison, we focus on improving productive efficiency via endogenous penalties for lying that arise as a result of failing trust.

The paper is organized as follows. Section 2 sets up the baseline model and characterizes the equilibrium without pre-trade communication. Section 3 contains the main analysis of the equilibrium with pre-trade communication, followed by the characterization of when such communication is welfare-enhancing in Section 4. Section 5 concludes.

# 2 Equilibrium without pre-trade communication

We start with an infinitely repeated version of a standard moral hazard model with two effort levels and two outcomes. To disentangle the effect of pure information sharing from the standard reputation effect, we abstract from adverse selection on the agent's type.

## 2.1 Baseline model

In each period $t \in \mathbb{N}$ of an infinite horizon model, a long-run seller privately exerts either high effort $h$ at a cost $c > 0$ or low effort $\ell$ at zero cost, to produce one unit of an experience good for sale. The quality of the produced item, denoted by $q_t$, is a random variable, independent across periods, that assumes a "good" value (i.e., $q_t = g$) with a high probability $h$ if high effort was exerted in that period but with a lower probability $\ell$ otherwise where $0 < \ell < h < 1$, and assumes a "bad" value (i.e., $q_t = b$) with the complementary probability. We assume that $q_t$ is unverifiable, albeit observable, *ex-post*, so no warranty contract is feasible on the realized quality. Note that $h$ and $\ell$ are used to denote both the effort levels and the associated probabilities of good quality being produced.

Multiple short-run buyers arrive afresh at the beginning of each period and leave at the end of the period. The item's quality captures the buyers' identical consumption value of the item, which we normalize as 1 and 0 for a good and bad quality item, respectively, i.e., $g = 1$ and $b = 0$. The buyers are risk-neutral and aim to maximize their expected consumption value net of price paid, but do not observe the realized quality of the item prior to purchase. Hence, due to competition, in each period the item is sold to one of the buyers at a price, $p_t$, that is equal to its expected quality based on the information shared by the buyers. The seller maximizes the expected discounted sum of her period surpluses which are the prices received net of effort cost. We assume that exerting effort is socially efficient, i.e.,

$$c \; < \; h - \ell. \tag{1}$$

The purchaser observes the item's true quality, $q_t$, and publicly reports it truthfully.[7] Therefore, a price and quality pair $(p_t, q_t) \in [0,1] \times \{g, b\}$ will be observed publicly after every period $t \in \mathbb{N}$. The seller is also informed of the effort she has exerted, but this private history is easily shown to have no effect on equilibrium payoffs.[8] Hence, we focus our attention on the strategies that depend only on publicly observable history. In order to exploit the self-generation idea of Abreu, Pearce and Stacchetti (1986), we allow for public randomization devices between periods, represented by an i.i.d. uniform random variable on [0,1] to be realized and observed publicly at the end of each period $t$.[9]

Let $H_t$ denote the (public) history at the beginning of period $t \in \mathbb{N}$, with $H_1 = \emptyset$ denoting the null history. Let $\mathcal{H}$ denote the set of all possible histories. A strategy of the seller is a mapping $e : \mathcal{H} \to [0,1]$, that specifies, conditional on each history $H_t \in \mathcal{H}$, a probability $e(H_t)$ with which the seller exerts high effort in period $t$.

A strategy $e$ determines a price $p(H_t|e)$ in each period $t$ as the expected quality of the item produced in that period given the history $H_t$, i.e., $p(H_t|e) = e(H_t)(h - \ell) + \ell$. Then, the seller's *normalized* expected payoff from an arbitrary strategy $\tilde{e}$ when consumers expect strategy $e$ from the seller is

$$v(\tilde{e}|e) \;=\; (1-\delta) \cdot \mathbf{E}\left[ \sum_{t=1}^{\infty} \delta^{t-1} \big( p(H_{t-1}|e) - \tilde{e}(H_t)c \big) \;\Big|\; \tilde{e} \right].$$

A strategy $e$ is a <u>perfect public equilibrium</u> (equilibrium, for short) of the baseline model described above if

$$v(e|e) \;\geq\; v(\tilde{e}|e) \quad \text{for every strategy } \tilde{e} \text{ of the seller.} \tag{2}$$

[7] This captures consumer feedback systems prevalent in online markets and is standard in related studies, e.g., Tadelis (1999), Mailath and Samuelson (2001) and Bar-Isaac (2003).

[8] Consider an equilibrium in which the seller behaves differently depending on her effort exerted in the past after some public histories. Upon reaching such a history, the seller's past effort has no direct influence on the continuation game and consequently, it is optimal for the seller with any private history of past effort to exert either effort because both must be optimal given that she behaves differently there depending on private history in the initial equilibrium. Thus, it constitutes an equilibrium for her to exert high effort with the average equilibrium probability regardless of private history for all public histories.

[9] This is purely for expositional convenience as our results hold without such randomization devices as shown in the Appendix (end of the proof of Proposition 1).

## 2.2 Characterization of equilibrium payoffs

For an arbitrary equilibrium, the seller's (normalized) payoff/value is

$$v \;=\; (1-\delta)(p_1 - e_1 c) + \delta\big[Prob(q_1 = g|e_1)\cdot v_g + Prob(q_1 = b|e_1)\cdot v_b\big] \tag{3}$$

where $e_1$ is the equilibrium probability with which the seller exerts $h$ in the first period, $p_1 = e_1(h-\ell)+\ell$ is the equilibrium price in the first period, and $v_q$ is the continuation value after the realized first period quality, $q_1$, is $q \in \{g, b\}$. Since the equilibrium price is no lower than $\ell$ in every period, any equilibrium value satisfies

$$v \;\geq\; \underline{v} := \ell.$$

In fact, $\ell$ is the minimum equilibrium value obtained in the "trivial equilibrium" in which the seller exerts $\ell$ in every period regardless of history.

Suppose there is a maximum equilibrium value of the seller, denoted by $v^*$. Then, the continuation values $v_q$ in (3) may be replicated by a public randomization among extreme continuation equilibria (cf. Abreu, Pearce and Stacchetti (1990) and Fudenberg and Levine (1994)): if the realized quality is $q \in \{g, b\}$ in the first period, the "maximum equilibrium" with the value $v^*$ ensues as the continuation equilibrium with probability $\rho_q$ while the trivial equilibrium with value $\underline{v}$ is triggered with probability $1-\rho_q$, where $\rho_g$ and $\rho_b$ satisfy

$$v_g = \rho_g v^* + (1-\rho_g)\underline{v} \quad \text{and} \quad v_b = \rho_b v^* + (1-\rho_b)\underline{v}. \tag{4}$$

With this representation, $v^*$ is the maximum self-generated value in the following sense: a value $v$ is *self-generated by a tuple* $(e_1, \rho_g, \rho_b) \in [0,1]^3$ if, assuming the first period price $p_1 = e_1(h-\ell)+\ell$ and the continuation values $v_q = \rho_q v + (1-\rho_q)\underline{v}$ for $q \in \{g, b\}$, $e_1$ is optimal for the seller in the first period and $v$ is her value. It is clear from the definition that a self-generated value is an equilibrium value (with public randomization).

If the maximum equilibrium value $v^*$ is self-generated by a tuple where $e_1 = 0$, the price is $\ell$ in every period because $v^*$ may be self-generated recursively as continuation values, and thus $v^* = \underline{v}$ entails. Below we focus on the cases where $v^*$ exceeds $\underline{v}$.

If a value $v > \underline{v}$ is self-generated by $(e_1, \rho_g, \rho_b)$, then $e_1 > 0$ and therefore the seller obtains the value $v$ by exerting $h$ in the first period:

$$v = (1-\delta)(p_1 - c) + \delta\big[h(\rho_g v + (1-\rho_g)\underline{v}) + (1-h)(\rho_b v + (1-\rho_b)\underline{v})\big]$$

where $p_1 = e_1(h-\ell)+\ell$. By rearranging, we express the self-generated value $v$ as a function of $p_1$, $\rho_g$ and $\rho_b$ as

$$v(p_1, \rho_g, \rho_b) := \frac{(1-\delta)(p_1 - \ell - c)}{1-\delta(h\rho_g + (1-h)\rho_b)} + \underline{v}. \tag{5}$$

The incentive compatibility that the seller does not benefit by exerting $\ell$ instead is

$$v \;\geq\; (1-\delta)p_1 + \delta[\ell(\rho_g v + (1-\rho_g)\underline{v}) + (1-\ell)(\rho_b v + (1-\rho_b)\underline{v})]$$

where $v = v(p_1, \rho_g, \rho_b)$. This is rearranged as

$$\delta(h-\ell)(\rho_g - \rho_b)\left(v(p_1,\rho_g,\rho_b) - \underline{v}\right) \;\geq\; (1-\delta)c \;\text{ with equality if }\; p_1 < h, \tag{ICh}$$

an inequality capturing that the future gain from exerting $h$ exceeds the current cost $c$. Conversely, any $v > \underline{v}$ is a self-generated value if $v = v(p_1, \rho_g, \rho_b)$ and satisfies the incentive compatibility condition (ICh) for some $(p_1, \rho_g, \rho_b) \in (\ell, h] \times [0,1]^2$.

Consequently, if the maximum self-generated value $v^*$ exceeds $\underline{v}$, it is the solution value to the linear program below:

$$v^* \;=\; \max_{\substack{0\leq\rho_g,\rho_b\leq 1\\ \ell<p_1\leq h}} \; v(p_1,\rho_g,\rho_b) \qquad \text{subject to (ICh).} \tag{P*}$$

As $v(p_1, \rho_g, \rho_b)$ increases in all its arguments while the LHS (left-hand side) of the inequality (ICh) decreases in $\rho_b$, the maximum value $v^*$ is obtained with $p_1 = h$ (hence the seller exerts high effort for sure), $\rho_g = 1$, and the largest $\rho_b$ subject to (ICh), which we denote by $\rho_b^*$. We obtain the solution values as

$$v^* = h - \frac{c(1-\ell)}{h-\ell} \;<\; h - c \;\text{ and }\; \rho_b^* = \frac{\delta(h-\ell)^2 - c(1-\delta\ell)}{\delta(h-\ell)^2 - c\delta(1-\ell)}. \tag{6}$$

It is straightforward to check that $0 \leq \rho_b^* \leq 1$ if and only if

$$c < c^* := \frac{(h-\ell)^2}{1-\ell} \qquad \text{and} \qquad \delta \geq \delta^*(c) := \frac{c}{(h-\ell)^2 + c\ell}. \tag{7}$$

If $c \geq c^*$ then the incentive compatibility (ICh) cannot hold for a legitimate tuple $(p_1, \rho_g, \rho_b)$, hence $v^* = \underline{v}$.

Observe that increasing $\ell$ and/or decreasing $h$ reduce both the value of $v^*$ and the threshold $c^*$. This ensues because, as the two effort levels generate closer quality distributions, the effort choice becomes harder to monitor *ex-post* by the delivered quality. In this sense, higher $\ell$ or lower $h$ worsens the moral hazard problem.

As asserted already, the maximum self-generated value $v^*$ obtained in the program (P*) is an equilibrium value if condition (7) holds. In addition, so is every value between this maximum and the lowest equilibrium value $\underline{v}$, as established in the next result.

**Proposition 1** *In the baseline model, the set of the seller's equilibrium values is an interval* $[\underline{v}, v^*]$ *if* $c < c^*$ *and* $\delta \in (\delta^*(c), 1)$*, and is a singleton* $\{\underline{v}\}$ *if* $c \geq c^*$.

*Proof.* In the Appendix. ■

Proposition 1 shows that the seller may be disciplined and efficiency improved by suitably rewarding her good past performance with higher continuation values, so long as she is patient enough and high effort is not too costly to exert. However, this is so only to a limited extent because the maximum equilibrium value, $v^*$ in (6), falls short of the socially efficient level, $h - c$, uniformly by $\frac{c(1-h)}{h-\ell}$. Note that the magnitude of inefficiency is proportional to the effort cost $c$ by a factor that increases in the severity of the moral hazard (higher $\ell$ or lower $h$). In addition, any disciplinary effect evaporates if $c \geq c^*$.

# 3 When pre-trade communication is possible

We now modify the baseline model of the previous section by postulating that in each period $t$, the seller observes a signal $s_t \in \{\mathbb{g}, \mathbb{b}\}$ regarding the realized quality of that period ($\mathbb{g}$ for good and $\mathbb{b}$ for bad), which is wrong with probability $\lambda < 1/2$.[10] After observing $s_t$, the seller publicly announces a message $m_t$ from a finite set $M$ of cheap-talk messages. Then, the item is sold to one of the buyers at a price, $p_t$, that is equal to its expected quality based on the information shared by the buyers, including $m_t$. As before, the purchaser observes the item's true quality, $q_t$, and publicly reports it truthfully. The game modified with communication as such is referred to as the "communication model."

Clearly, the maximum value $v^*$ obtained in the previous section can be replicated in the communication model via the so-called "babbling" announcement. We show that there are other equilibria in which communication enhances efficiency, achieving full efficiency asymptotically as $\lambda \to 0$. This means that communication can improve the market's ability to reward more efficient production and thereby, the social welfare.[11]

## 3.1 Preliminaries

Formally, a history $H_t$ is defined as before except that it now includes the record of all past messages along with prices and qualities. A *strategy* of the seller is a pair $(e, a)$ of mappings where the effort strategy, $e : \mathcal{H} \to [0, 1]$, specifies for each history a probability with which the seller exerts high effort and the announcement strategy, $a : \mathcal{H} \times \{h, \ell\} \times \{\mathbb{g}, \mathbb{b}\} \to \Delta(M)$, specifies a mixed strategy of sending a message conditional on the effort exerted and the signal observed, as well as history. We use $e_1$ and $a_1$ to denote the first period strategies.

[10] That is, $s_t = \mathbb{g}$ (resp. $\mathbb{b}$) with probability $1 - \lambda$ conditional on the realized quality $q_t = g$ (resp. $b$).

[11] Notice that as history does not affect the continuation game in this model, the seller exerts $\ell$ and no communication takes place in the unique Markov perfect equilibrium. This contrasts with the case of adverse selection where communication may improve efficiency even when equilibria are restricted to be Markov (see Jullien and Park, 2014).

The seller's expected payoff and equilibrium are defined analogously to the baseline model of no communication, accounting for the information transmitted by messages.

The seller's announcement is about her superior information on the quality produced owing to her knowledge of the effort exerted and the signal observed. Now that the continuation values may depend on the message as well as the quality delivered, her optimal announcement varies depending on the posterior probability she believes that the actual quality is good conditional on the signal observed and the effort exerted. We let $\pi_s$ denote this probability conditional on observing a signal $s \in \{\mathbb{g}, \mathbb{b}\}$ after exerting $h$:

$$\pi_{\mathbb{g}} = \frac{h(1-\lambda)}{h(1-\lambda)+(1-h)\lambda} \quad \text{and} \quad \pi_{\mathbb{b}} = \frac{h\lambda}{h\lambda+(1-h)(1-\lambda)} < \pi_{\mathbb{g}}.$$

Similarly, let $\pi'_s$ denote that after exerting $\ell$:

$$\pi'_{\mathbb{g}} = \frac{\ell(1-\lambda)}{\ell(1-\lambda)+(1-\ell)\lambda} < \pi_{\mathbb{g}} \quad \text{and} \quad \pi'_{\mathbb{b}} = \frac{\ell\lambda}{\ell\lambda+(1-\ell)(1-\lambda)} < \pi_{\mathbb{b}}.$$

If the seller's signal is precise enough, the seller is more optimistic about quality when she observes a good signal after low effort than when she observes a bad signal after high effort. More precisely,

$$\pi_{\mathbb{b}} < \pi'_{\mathbb{g}} \iff \lambda < \widetilde{\lambda} := \frac{\sqrt{(1-h)\ell}}{\sqrt{(1-h)\ell}+\sqrt{h(1-\ell)}}. \tag{8}$$

Condition (8) means that a good signal ($s = \mathbb{g}$) indicates a higher average quality than a bad signal ($s = \mathbb{b}$) irrespective of the effort exerted.

Notice that $\underline{v} = \ell$ is the minimum equilibrium value in the modified game as well, which is obtained by the trivial equilibrium in which the seller always exerts $\ell$ and babbles. We are interested in the maximum value of the seller obtainable in equilibrium with communication, which we denote by $\overline{v}$. As before, for any equilibrium, the continuation value after message $m \in M$ and realized quality $q \in \{g, b\}$ of period 1, denoted by $v_{mq}$, can be replicated by the public randomization probability $x_{mq} \in [0, 1]$ such that

$$v_{mq} = x_{mq}\overline{v} + (1 - x_{mq})\underline{v}. \tag{9}$$

For the same reason as in the case of no communication, $\overline{v} = \underline{v}$ ensues if the probability of exerting high effort is nil in the first period (i.e., $e_1 = 0$) of the maximum equilibrium. We analyze below the environments where $\overline{v} > \underline{v}$, which implies that $e_1 > 0$ in the maximum equilibrium.

Observe that $\overline{v}$ is "self-generated" in the sense described earlier: a seller's value $v$ is self-generated by a period strategy $(e_1, a_1)$ and randomization probabilities $(x_{mq})_{m\in M, q\in\{g,b\}}$ if, assuming the first period price $p_1(m) = \mathbf{E}(q_1|e_1, a_1, m)$ and the continuation values

$v_{mq} = x_{mq}v + (1 - x_{mq})\underline{v}$ for $m \in M$ and $q \in \{g, b\}$, $(e_1, a_1)$ is optimal for the seller in the first period and $v$ is her value. As before, we characterize $\overline{v}$ by identifying the maximum self-generated value. Specifically, we first identify the maximum self-generated value $\overline{v}_F$ obtained by the period strategy of exerting $h$ and reporting the signal truthfully, which we call "faithfully self-generated" values. We then show (in Appendix) that there is no higher self-generated value with meaningful communication, that is, untruthful reporting and/or mixing $h$ and $\ell$ do not self-generate a higher value. Subsequently, we establish that every value in $[\underline{v}, \overline{v}_F]$ is an equilibrium value of the communication model if $\delta$ is large enough (Proposition 3).

## 3.2 Faithfully self-generated values (FSGV)

We start by analyzing faithfully self-generated values (FSGV) for which the seller adopts the following "faithful strategy" every period until the trivial equilibrium is triggered: exert $h$ for sure and report the observed signal truthfully, that is, report $m = G$ upon observing a good signal $s = \mathbb{g}$ and $m = B$ upon observing a bad signal $s = \mathbb{b}$.[12] Then, the expected qualities, hence the prices induced by messages $m = G$ and $m = B$ are, respectively,

$$p_G = \pi_{\mathbb{g}} > h \quad \text{and} \quad p_B = \pi_{\mathbb{b}} < h \tag{10}$$

until the trivial equilibrium is triggered by public randomization.

For an arbitrary FSGV, say $v_F$, we denote by $\mathbf{x} = (x_{Bg}, x_{Gg}, x_{Bb}, x_{Gb}) \in [0, 1]^4$ the vector of randomization probabilities that the seller continues with the faithful strategy (i.e., the trivial equilibrium is *not* triggered) subsequent to sending a message $m \in \{B, G\}$ and delivering a quality $q \in \{g, b\}$ (as reported by the purchaser) in the first period. Then, her continuation value upon delivering quality $q \in \{g, b\}$ is $v_F$ with probability $\rho_q$ defined as

$$\rho_g := (1 - \lambda)x_{Gg} + \lambda x_{Bg} \quad \text{and} \quad \rho_b := (1 - \lambda)x_{Bb} + \lambda x_{Gb}. \tag{11}$$

Hence, $v_F$ satisfies the Bellman equation:

$$v_F = (1 - \delta)\big[p_B + (h(1 - \lambda) + (1 - h)\lambda)(p_G - p_B) - c\big] + \delta\big[(h\rho_g + (1 - h)\rho_b)(v_F - \underline{v}) + \underline{v}\big].$$

Since the expected price in the first period is $h$ (the expected quality from high effort), this equation can be rearranged to express $v_F$ in terms of $\mathbf{x}$, in particular $\rho_g$ and $\rho_b$, as

$$v_F(\mathbf{x}) := \frac{(1 - \delta)(h - \ell - c)}{1 - \delta(h\rho_g + (1 - h)\rho_b)} + \underline{v}. \tag{12}$$

[12] Truthful reporting means that the sets of messages sent after signals $\mathbb{g}$ and $\mathbb{b}$ in equilibrium are disjoint. Hence, we may restrict the message space to $\{G, B\}$ without loss of generality.

Note that this is the same form as the objective function in the program (P*) without communication when $p_1 = h$. Hence, communication will raise the maximum self-generated value only if it can motivate the seller to exert high effort with a lower probability of triggering the trivial equilibrium.[13]

We now spell out the optimality conditions that must hold for any FSGV $v_F(\mathbf{x})$ and characterize the maximum FSGV.

## 3.3 Incentive compatibility conditions

Consider a value $v_F(\mathbf{x})$ for a vector of randomization probabilities $\mathbf{x} = (x_{Bg}, x_{Gg}, x_{Bb}, x_{Gb}) \in [0,1]^4$. For $v_F(\mathbf{x})$ to constitute a FSGV, the seller must find the faithful strategy optimal until the trivial equilibrium kicks in.

To facilitate exposition, we proceed presuming that a deviating seller who has exerted $\ell$ (an $\ell$-seller, for short) follows a truthful announcement strategy (and consider other possibilities in due course). Then, the condition for $h$ to be the optimal effort choice is

$$v_F(\mathbf{x}) \geq (1-\delta)\big[p_B + (\ell(1-\lambda) + (1-\ell)\lambda)\,(p_G - p_B)\big] + \delta\big[(\ell\rho_g + (1-\ell)\rho_b)(v_F(\mathbf{x}) - \underline{v}) + \underline{v}\big].$$

This is rewritten as

$$\delta(h-\ell)(\rho_g - \rho_b)(v_F(\mathbf{x}) - \underline{v}) \;\geq\; (1-\delta)\big[c - (h-\ell)(1-2\lambda)(p_G - p_B)\big] \qquad (\mathrm{ICh}_{GB})$$

where the LHS is the expected benefit and the RHS is the net cost of exerting $h$. Note that choosing $h$ entails two effects: first, a good signal is more likely, boosting the current price (the second term of the RHS); second, a good quality is more likely, enhancing the continuation values (the LHS).

Condition ($\mathrm{ICh}_{GB}$) is the counterpart of the optimality condition (ICh) for exerting $h$ without communication analyzed in the previous section. Comparing the RHS of the two conditions reveals that communication reduces the cost of exerting $h$ by $(h-\ell)(1-2\lambda)(p_G - p_B)$. This is the expected gain in current price from exerting $h$ (as opposed to $\ell$), given that the price now reflects the *interim* information on quality communicated by the seller, thus is more aligned with the value created. This gain increases as the seller's information becomes more precise, reaching the full contribution of high effort, $h-\ell$, when her information is perfect. As such, pre-trade communication creates a differential in the expected current revenue between exerting high and exerting low effort.

Such short-term incentives complement the conventional long-term incentives provided via continuation value differentials and thereby, relax the constraint for inducing high effort from (ICh) to ($\mathrm{ICh}_{GB}$). For such easing of constraint, it is essential that the seller's *interim*

[13]This reflects the specific feature that the expected price is the same regardless of communication conditional on the effort in our model.

information on quality is from over and above her knowledge of the exerted effort. That is, the seller possesses *interim* information on expected quality from knowing the effort level she exerted. However, communication of this information is redundant and ineffective because the effort level, correctly anticipated, is fully reflected in the equilibrium price already. In contrast, communication on signal conveys new information, lending additional scope for the market to provide incentives.

However, such gain from communication comes with a cost in the form of extra constraints elaborated below. First, truthful announcement must be incentivized. Second, communication opens up multiple ways for the seller to deviate after shirking.

We start with the condition for truthful announcement: a seller who follows the faithful strategy and has exerted effort $h$ (an $h$-seller, for short) must prefer to send the message $B$ (resp. $G$) upon observing a bad signal (resp. a good signal). This truth-telling condition for an $h$-seller after $s = \mathbb{b}$ is

$$\begin{aligned} & (1-\delta)p_B + \delta\big([\pi_{\mathbb{b}} x_{Bg} + (1-\pi_{\mathbb{b}})x_{Bb}](v_F(\mathbf{x}) - \underline{v}) + \underline{v}\big) \\ \geq \;\; & (1-\delta)p_G + \delta\big([\pi_{\mathbb{b}} x_{Gg} + (1-\pi_{\mathbb{b}})x_{Gb}](v_F(\mathbf{x}) - \underline{v}) + \underline{v}\big), \end{aligned}$$

which can be rewritten as

$$\delta\left(\pi_{\mathbb{b}}\Delta_g + (1-\pi_{\mathbb{b}})\Delta_b\right)(v_F(\mathbf{x}) - \underline{v}) \;\geq\; (1-\delta)(p_G - p_B) \qquad \text{(ICB)}$$

$$\text{where} \;\; \Delta_q := x_{Bq} - x_{Gq} \;\; \text{for} \;\; q \in \{b, g\}.$$

We interpret this condition as follows: By announcing $G$ rather than $B$, the seller would garner a higher price but risks an increased likelihood $\Delta_q$ of losing trust and triggering the trivial equilibrium, depending on the actual quality. When the signal is bad, this risk must be large enough for an $h$-seller not to mislead the market for a higher price as captured by (ICB). When the signal is good, on the other hand, this risk must be small enough for an $h$-seller to opt to announce $G$ and get the high price $p_G$.[14] This incentive compatibility for an $h$-seller to announce $s = \mathbb{g}$ truthfully is:

$$(1-\delta)(p_G - p_B) \;\geq\; \delta\left(\pi_{\mathbb{g}}\Delta_g + (1-\pi_{\mathbb{g}})\Delta_b\right)(v_F(\mathbf{x}) - \underline{v}) \qquad \text{(ICG)}$$

Intuitively, announcing $G$ should pose a greater risk of triggering the trivial equilibrium when the quality turns out to be bad than when it turns out to be good. Indeed, the two conditions (ICB) and (ICG) imply that

$$\Delta_b \;\geq\; \Delta_g. \qquad \text{(D)}$$

[14]Hence, the game may be viewed alternatively as one in which the seller posts a price $p_B$ or $p_G$ until the trivial equilibrium gets triggered by a public randomization contingently on the posted price and the realized quality. A FSGV corresponds to a "truthful" equilibrium in which the seller finds it optimal to exert $h$ and post a price according to the observed signal until the trivial equilibrium is triggered. We do not adopt this approach because the game with posted prices has other equilibrium prices due to price signalling.

Next, returning to the incentive compatibility condition for exerting $h$, notice that the optimal announcement of an $\ell$-seller is determined by comparing the continuation values from announcing $m = G$ and $m = B$, which are the same formulae as (ICB) and (ICG) above with $\pi_s$ replaced by $\pi'_s$. Thus, (ICB) and (D) imply that an $\ell$-seller would truthfully announce $B$ after $s = \mathbb{b}$ because $\pi'_{\mathbb{b}} < \pi_{\mathbb{b}}$; but she may or may not announce $s = \mathbb{g}$ truthfully depending on the value of $\pi'_{\mathbb{g}}$. Hence, the condition for $h$ to be the optimal effort choice dictates, in addition to the condition (ICh$_{GB}$) earlier that an $\ell$-seller is not better-off with announcing both signals truthfully, that she is not better-off with announcing $B$ after both signals, either, i.e.,

$$(1 - \delta(\ell x_{Bg} + (1-\ell)x_{Bb}))(v_F(\mathbf{x}) - \underline{v}) \;\geq\; (1-\delta)(p_B - \ell) \qquad \text{(ICh}_{BB}\text{)}$$

where the LHS is the loss of discounted continuation value and the RHS is the initial period payoff gain from the deviation with pooled announcement.

To recap, the value $v_F(\mathbf{x})$ defined in (12) is a FSGV if and only if the four incentive compatibility conditions hold, namely, (ICB), (ICG), (ICh$_{GB}$) and (ICh$_{BB}$). In this case, we say that the FSGV is "supported by" the configuration $\mathbf{x}$.

Typically, not all four constraints bind at the maximum FSGV. Intuition suggests that it is more costly to induce truthful announcement of a bad signal than that of a good signal because the former requires compensating the seller for a low price. In addition, truth-telling is more likely to be optimal even after exerting $\ell$ as the signal becomes more precise, rendering (ICh$_{GB}$) to be the relevant constraint for exerting $h$, because then the signal primarily governs the future regardless of the effort exerted. The next lemma formalize these intuitions:

**Lemma 1** *The maximum FSGV, which we denote by $\overline{v}_F$ if exists, is the solution value to the relaxed linear program:*

$$\overline{v}_F \;=\; \max_{\mathbf{x}\in[0,1]^4} \; v_F(\mathbf{x}) \qquad (\overline{P})$$

*subject to (ICh$_{GB}$), (ICh$_{BB}$), (ICB) and (D).*

*Moreover, $\overline{v}_F$ is supported by a configuration $\mathbf{x} \in [0,1]^4$ that binds (ICB), but leaves (ICh$_{BB}$) slack if $\lambda < \widetilde{\lambda}$ and leaves (ICh$_{GB}$) slack if $\lambda > \widetilde{\lambda}$.*

*Proof.* In the Appendix. ■

As in standard pure adverse selection problems, a sorting condition holds for announcement that allows us to abstract from the incentive constraint for the seller with most favourable information, (ICG), so long as the monotonicity condition is satisfied. Hence, it turns out that every FSGV can be supported by a configuration that binds (ICB), the condition for truthful announcement of a bad signal. Recall that binding (ICB) means that the seller finds the two messages equivalent when $\pi_{\mathbb{b}}$ is her posterior on the quality of the produced item, in which case (ICG) is equivalent to and thus is replaced by (D) in $(\overline{P})$. Since the message $G$ is relatively more attractive than $B$ as her posterior is higher, an $\ell$-seller would prefer to announce a good signal truthfully when the signal is precise enough so that $\pi'_{\mathbb{g}} > \pi_{\mathbb{b}}$, i.e. $\lambda < \widetilde{\lambda}$, in which case (IChGB) is the relevant constraint for exerting $h$ and (ICh$_{BB}$) is redundant at the solution; and the converse holds otherwise.

One difference from pure standard adverse selection problems, however, is that achieving the maximum surplus may but needs not require binding the incentive constraints for truthful announcement. This is because the reputational rent needed for inducing high effort may be sufficient for inducing truthful announcement as well.

Given that communication eases some constraint but creates new constraints in an extra dimension, *a priori* it is unclear when effective communication is possible, let alone whether communication will improve efficiency or not. We answer these questions below.

## 3.4 Maximum FSGV and efficiency

As will be shown later, whenever an equilibrium exists in which the seller's announcement differs from babbling, so does a FSGV. Hence, characterizing when effective communication is possible amounts to characterizing when a FSGV exists and thus, when a configuration $\mathbf{x}$ exists that supports $\overline{v}_F$ in the manner described in Lemma 1, in particular, with (ICB) binding. Given that the announcement strategy of an $\ell$-seller depends on whether her signal is precise enough so that $\pi'_{\mathbb{g}} > \pi_{\mathbb{b}}$, the characterizations below also depend on whether $\lambda < \widetilde{\lambda}$ or not.

Since the "punishment" payoff $\underline{v}$ is independent of the cost of effort $c$ while the value $v_F(\mathbf{x})$ decreases in $c$, the higher $c$ is the harder it is to sustain the faithful strategy. This is reflected in the conditions (ICB), (ICh$_{GB}$) and (ICh$_{BB}$) becoming harder to satisfy as $c$ increases. Thus, if a configuration $\mathbf{x}$ satisfies all constraints of $(\overline{P})$ for some level of $c$ and thus supports a FSGV, so it does for all lower levels of $c$. This means that a FSGV (hence, the maximum FSGV) exists for all $c$ below a threshold level. We show in the Appendix that this threshold converges, as $\delta$ tends to 1, to[15]

$$\overline{c}\,(\lambda) := \begin{cases} \min\left\{c^* + \frac{(1-2\lambda)(h-\ell)(1-h)(p_G-p_B)}{1-\ell},\ h-\ell-\frac{\lambda(1-h)(p_G-p_B)}{1-p_B}\right\} & \text{if } \lambda \leq \widetilde{\lambda}, \\ h-\ell-\frac{(1-h)(p_B-\ell)}{1-\ell}-\frac{\lambda(1-h)(p_G-p_B)}{1-p_B} & \text{if } \lambda \geq \widetilde{\lambda}. \end{cases} \tag{13}$$

Therefore, a FSGV exists for large enough $\delta$ if and only if $c < \overline{c}(\lambda)$. Note that $\overline{c}(0) = h-\ell$ and $\overline{c}(1/2) = c^* < h-\ell$ but $\overline{c}(\lambda)$ is not monotone as will be discussed in the sequel.

To characterize the maximum FSGV when $c < \overline{c}(\lambda)$ and $\delta$ is large enough, it proves useful to know which other constraints bind at the solution $\mathbf{x}$ to $(\overline{P})$ that binds (ICB).

[15]Note that $\overline{c}(\widetilde{\lambda}) = c^* + \frac{(1-2\widetilde{\lambda})(h-\ell)(1-h)(p_G-p_B)}{1-\ell} = h-\ell-\frac{(1-h)(p_B-\ell)}{1-\ell}-\frac{\widetilde{\lambda}(1-h)(p_G-p_B)}{1-p_B} > c^*$.

Observe from (ICB) that $\overline{v}_F - \underline{v}$ exceeds a minimal rent $(1-\delta)(p_G - p_B)$ independently of the effort cost $c$. When $c$ is small enough, this minimal rent should be sufficient to incentivize the seller to exert high effort and consequently, the maximum FSGV should be achieved without binding the incentive compatibility conditions for inducing high effort, (ICh$_{GB}$) or (ICh$_{BB}$). We show in the Appendix that this is indeed the case for large $\delta$ if $c < \widehat{c}(\lambda)$ where

$$\widehat{c}(\lambda) := \begin{cases} (h-\ell)(p_G - p_B)\left(1 - 2\lambda + \frac{\lambda}{1-p_B}\right) & \text{if } \lambda \le \widetilde{\lambda}, \\ h - p_B - \frac{\lambda(1-h)(p_G-p_B)}{1-p_B} & \text{if } \lambda \ge \widetilde{\lambda}. \end{cases} \tag{14}$$

Note that $\widehat{c}(0) = h - \ell$ and $\widehat{c}(1/2) = 0$.

Observe that $\widehat{c}(\lambda) < \overline{c}(\lambda)$ for $\lambda > \widetilde{\lambda}$ but not necessarily for $\lambda < \widetilde{\lambda}$ and both thresholds depend on which incentive compatibility constraint for inducing $h$ is relevant. As such, depending on whether $\lambda$ is below or above $\widetilde{\lambda}$, the details differ on how the maximum efficiency may be achieved. Yet, a similar general insight prevails that the most effective way is to reward good quality and truthful announcement of bad signal, as stated below.

**Proposition 2** *A FSGV exists for large enough $\delta$ if and only if $c \in (0, \overline{c}(\lambda))$. In this case, the maximum FSGV, $\overline{v}_F$, is supported by a configuration $\bar{\mathbf{x}} = (\bar{x}_{Bg}, \bar{x}_{Gg}, \bar{x}_{Bb}, \bar{x}_{Gb}) \in [0,1]^4$ such that*

*(i) $\bar{x}_{Bg} = \bar{x}_{Gg} = 1$ (i.e., $\bar{\rho}_g = 1$) and moreover, either*

*(ii) $\bar{x}_{Bb} = 1$ and $\bar{x}_{Gb} < 1$ that binds the truth-telling constraint (ICB) if $c \le \widehat{c}(\lambda)$, in which case*

$$\overline{v}_F = h - c - \frac{(1-h)h\lambda(1-2\lambda)}{(1-\lambda)(\lambda + h(1-2\lambda))}, \quad \text{or} \tag{15}$$

*(ii′) $\bar{x}_{Bb} < 1$ and $\bar{x}_{Gb} < 1$ that bind both (ICB) and the relevant constraint for inducing $h$, i.e., (ICh$_{GB}$) for $\lambda \le \widetilde{\lambda}$ and (ICh$_{BB}$) for $\lambda > \widetilde{\lambda}$, if $c > \widehat{c}(\lambda)$.*

*In either case, $\bar{\rho}_b \to 1$ as $\delta \to 1$.*

*Proof.* In the Appendix. ■

Hence, unless the cost $c$ is too large the seller can be incentivized to exert high effort and truthfully disclose the *interim* information on quality prior to trade for an indefinite length of time. This is achieved via rewarding the seller by never triggering the trivial equilibrium provided that the delivered quality is good, and also provided that bad quality is disclosed when $c \le \widehat{c}(\lambda)$. In this case, as the maximum surplus of $h - c$ is accrued until the trivial equilibrium is triggered with a probability no higher than $(1-h)\lambda$ in each period, a lower bound for the maximum FSGV, $\overline{v}_F$, is obtained as

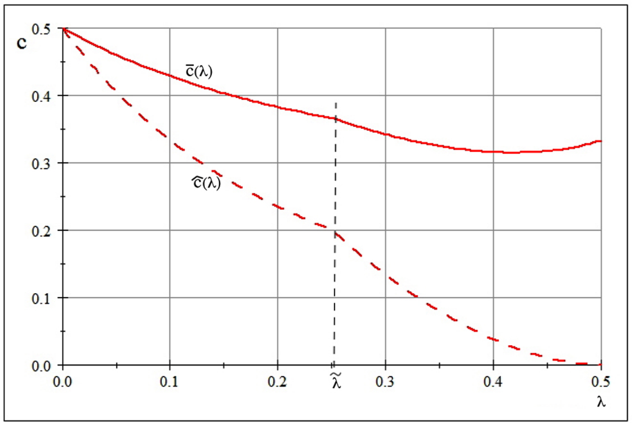
$$\overline{v}_F \ \ge \ \frac{(1-\delta)(h-c)}{1-\delta(1-(1-h)\lambda)} \quad \to \quad h - c \ \text{ as } \ \lambda \to 0.$$

Since $\widehat{c}(\lambda) \to h - \ell$ as $\lambda \to 0$, this means that full efficiency is achieved for all $c < h - \ell$ via pre-trade communication as the observation error $\lambda$ vanishes, which is also evident from (15). This is the case for all large enough $\delta$ (rather than asymptotically as $\delta \to 1$).

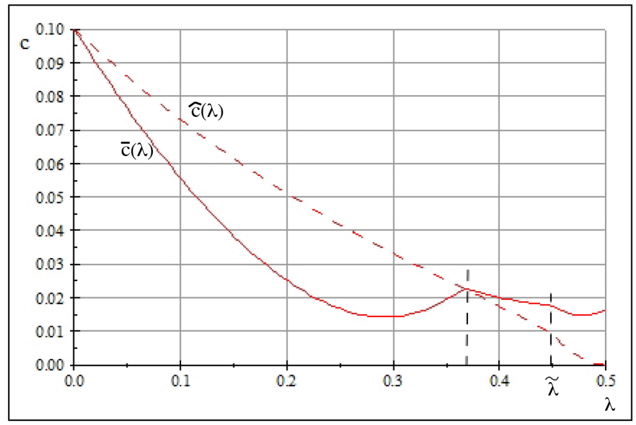
Observe that $\overline{v}_F$ in (15) is convex in $\lambda$ reaching a minimum at $\lambda = \check{\lambda} := \frac{\sqrt{h}}{1+2\sqrt{h}} < 1/3$. This reflects the dual effect of noisier signal on the truth-telling incentives: it discourages truth-telling because lying is more likely to go undetected (which is the dominant effect for low $\lambda$) but also encourages it by reducing the price differential to be exploited by lying. Consequently, the maximum efficiency changes non-monotonically as the signal gets noisier: $\overline{v}_F$ decreases in $\lambda < \check{\lambda}$ as truth-telling becomes harder to induce, and increases in $\lambda > \check{\lambda}$ so long as $c \leq \widehat{c}(\lambda)$. Note that the magnitude of inefficiency is independent of $c$. This ensues because, unlike in the case of no communication in the previous section, the solution is determined by the binding constraint of announcement (ICB) and the effort cost is sunk at the point of announcement.

The formulae of $\overline{v}_F$ for $c > \widehat{c}(\lambda)$ are lengthier and less intuitive to interpret, so they are provided in the Appendix. As they are obtained from the solution to binding (ICB) in conjunction with the corresponding constraint for efficient effort, the aforesaid non-monotonic effects of $\lambda$ on truth-telling incentives continue to have interesting implications as explained below. It may be worth noting that the solution needs not be unique in this case. In particular, when the signal is precise enough ($\lambda < \widetilde{\lambda}$) and the incentive compatibility for effort is binding ($c > \widehat{c}(\lambda)$), there exist other solutions where (ICB) is not binding and/or truthful announcement of bad signal is not penalized ($x_{Bb} = 1$).

We illustrate the two thresholds $\overline{c}(\lambda)$ and $\widehat{c}(\lambda)$, in Figure 1-(a) for the case that $h - \ell$ is relatively large with $(h, \ell) = (0.75, 0.25)$ and in Figure 1-(b) for the case that it is relatively small with $(h, \ell) = (0.5, 0.4)$. In both diagrams, the red curve represents $\overline{c}(\lambda)$ below which a FGSV exists and the dotted curve represents $\widehat{c}(\lambda)$. Both curves are continuous, but with kinks. It is clear from (13) and (14) that the curve $\widehat{c}(\lambda)$ has only one kink at $\lambda = \widetilde{\lambda}$ where $\overline{c}(\lambda)$ also has a kink. In Figure 1-(a), $\overline{c}(\lambda)$ has no other kink because the first argument of the min operator in (13) is smaller for all $\lambda < \widetilde{\lambda}$. In Figure 1-(b), on the other hand, $\overline{c}(\lambda)$ has another kink at a value of $\lambda$ ($< \widetilde{\lambda}$) where the two arguments of the min operator coincide. The segments of $\lambda$ in which $\overline{c}(\lambda)$ is smooth but is not monotonic stems from the non-monotonic effects of $\lambda$ on incentives for truth-telling mentioned above. For low $c$, a FSGV exists for all $\lambda < 1/2$. In this case, the price differential is small when the signal is very noisy as noted above, but so is the risk differential $\Delta_b$ upon delivering a bad quality. A deviating $\ell$-seller would announce $B$ even when the signal is good and the incentive constraint for effort is binding as $c > \widehat{c}(\lambda) \approx 0$. It may be worth noting from Figure 1-(b) that when $\ell$ is close to $h$ the incentive constraint for effort is not binding at the maximum FGSV for a wide range of signal precision because $c < \overline{c}(\lambda) < \widehat{c}(\lambda)$.



(a) $\overline{c}(\lambda)$ and $\widehat{c}(\lambda)$ for large gap $h-\ell$



(b) $\overline{c}(\lambda)$ and $\widehat{c}(\lambda)$ for small gap $h-\ell$

**Figure 1**

Clearly, there are self-generated values associated with seller's strategies other than the faithful strategy, e.g., those obtained in the previous section (without communication) can be replicated via babbling as mentioned earlier. As we show in the Appendix, however, $\overline{v}_F$ is the upper bound of all self-generated values supported by non-babbling strategies if $\delta$ is large enough. Thus, the higher value of $\overline{v}_F$ and $v^*$ is the maximum self-generated value when pre-trade communication is possible and moreover, any value between $\underline{v}$ and this maximum constitutes an equilibrium value by the same logic used for the case without communication. We now state the results on possible equilibrium values and efficiency.

**Proposition 3** *In the communication model, the set of all equilibrium values is the interval* $[\underline{v}, \max\{v^*, \overline{v}_F\}]$ *for large enough* $\delta$. *Moreover, the value* $\overline{v}_F$ *converges to* $h-c$ *as* $\lambda \to 0$.

*Proof.* In the Appendix. ■

Therefore, if efficiency can be improved through rewarding a seller's past performance via continuation values (i.e., $c < c^*$), it can be further enhanced by allowing for pre-trade communication if and only if it can be done via a faithful strategy. If $c \geq c^*$, communication is clearly beneficial if a FSGV exists at all. It has been shown above that communication is beneficial if $\lambda$ is low enough. In the next section, we fully characterize the environments in which communication enhances welfare.

# 4 When is communication beneficial?

We showed in the previous section that the moral hazard problem can be fully resolved through pre-trade communication if the seller observes the quality without error and is patient enough. In this section, we clarify how widespread is such welfare-enhancing effect of communication by delineating the parameter values for which the maximum value achievable for a patient seller is higher with pre-trade communication than without. We proceed by comparing $\overline{v}_F$, the maximum equilibrium value with faithful communication, with the corresponding value $v^*$ without communication obtained in Section 2 for large enough $\delta$.

When $c \geq c^*$ so that high effort cannot be induced without communication at all, communication is clearly beneficial so long as a FSGV exists, i.e., if $c < \bar{c}(\lambda)$. We will elaborate later how large $\bar{c}(\lambda)$ is.

Consider the case that $c < c^*$ so that $v^* > \underline{v}$, i.e., high effort can be sustained without communication to some extent, in particular, with probabilities $\rho_g^* = 1$ and $\rho_b^* < 1$ after the seller delivered quality $q \in \{g, b\}$ in the preceding period. In this case, communication enhances efficiency if high effort can be induced with a larger $\rho_b$, or equivalently, with a lower probability of triggering punishment after delivery of bad quality. This is feasible in principle because truthful communication eases the incentive constraint for inducing high effort from (ICh) to (ICh$_{GB}$) by aligning the price with the realized quality. But, inducing truthful communication adds two extra constraints (ICB) and (ICh$_{BB}$) as explained earlier, each of which turns out to be a potential barrier that limits the benefits of communication in certain environments as elaborated below.

To derive some intuition on when the optimal configuration $\bar{\mathbf{x}}$ in Proposition 2 generates $\overline{v}_F$ that exceeds $v^*$, let us set $x_{Gg} = x_{Bg} = 1$ so that $\rho_g = 1$, and look for the maximum $v_F(\mathbf{x})$ subject only to (ICB) and (D). Since the LHS of (ICB) increases in $x_{Bb}$ and decreases in $x_{Gb}$, to maximize $v_F(\mathbf{x})$ we need $x_{Bb} = 1$ and $x_{Gb} = \hat{x}_{Gb}$ that binds (ICB). For $\overline{v}_F > v^*$, therefore, it is necessary that $1 - \lambda + \lambda \hat{x}_{Gb} > \rho_b^*$ for all large enough $\delta$, which is verified to be the case if and only if

$$c \ > \ \underline{c}(\lambda) := \lambda(h - \ell)\Big(\frac{p_G - p_B}{1 - p_B}\Big). \tag{16}$$

To understand what goes on when $c < \underline{c}(\lambda)$, recall that overstating signal (announcing $G$ upon observing $s = \mathbb{b}$) is discouraged by the spread $\Delta_b = x_{Bb} - x_{Gb}$ of the probability to continue with the faithful strategy between truthful announcement of bad quality and overstatement. In particular, (ICB) requires that this spread $\Delta_b$ be bounded away from 0 because the future loss from overstating a bad signal must overshadow the current gain in price differential, $p_G - p_B$. When $c$ is very small, the probability $\rho_b^*$ is very close to 1, leaving little scope to push $x_{Bb}$ above $\rho_b^*$ to have enough spread $\Delta_b$. Thus, truth-telling must be induced by decreasing $x_{Gb}$ significantly, sacrificing efficiency. This is the barrier stemming from the truth-telling constraint (ICB).

Hence, for communication to enhance efficiency it is necessary that $\rho_b^*$ is not too close to 1, i.e., $c$ is not too small in the sense of (16). In fact, provided that a FSGV exists, i.e.,

$c \leq \bar{c}(\lambda)$, (16) is also sufficient (for communication to be beneficial) if either

$$\lambda < \widetilde{\lambda}, \quad \text{or} \quad \lambda > \widetilde{\lambda} \text{ and } c \leq \widehat{c}(\lambda). \tag{17}$$

To see this, consider the configuration $\mathbf{x}^* = (1, 1, 1, x^*_{Gb})$ where $1 - \lambda + \lambda x^*_{Gb} = \rho^*_b$ so that $v_F(\mathbf{x}^*) = v^*$. If (16) holds, (ICB) and (D) are slack at $\mathbf{x}^*$. Thus, $\overline{v}_F > v^*$ ensues if the incentive constraint for inducing $h$ is also slack at $\mathbf{x}^*$. This is indeed the case if $\lambda < \widetilde{\lambda}$ because the relevant constraint is eased from (ICh) to (ICh$_{GB}$) by aligning the price with realized quality as noted earlier. When $\lambda > \widetilde{\lambda}$ so that the relevant condition is (ICh$_{BB}$), $\overline{v}_F > v^*$ obtains if $c \leq \widehat{c}(\lambda)$ because then $\overline{v}_F$ is supported by the configuration $\hat{\mathbf{x}} = (1, 1, 1, \hat{x}_{Gb})$ that binds (ICB) according to Proposition 2,[16] and $v_F(\hat{\mathbf{x}}) > v_F(\mathbf{x}^*)$ by (16). This is also directly verified by comparing $\bar{v}_F$ in (15) with $v^*$ in (6).

In the remaining case that $\lambda > \widetilde{\lambda}$ and $c > \widehat{c}(\lambda)$, the second barrier comes into play. In particular, at the solution configuration $\bar{\mathbf{x}}$ described in Proposition 2 where $\rho_g = 1$, the truth-telling constraint (ICB) reduces to

$$\Delta_b \; \geq \; \frac{p_G - p_B}{(1 - p_B)(h - \ell - c)} \left( \frac{1}{\delta} - h - (1 - h)\rho_b \right), \tag{18}$$

whereas the relevant incentive constraint for inducing $h$, (ICh$_{BB}$), can be rewritten as (using $x_{Bb} = \rho_b + \lambda \Delta_b$)

$$\lambda \Delta_b \leq \left( \frac{h - p_B - c}{1 - \ell} \right) \frac{1 - \delta}{\delta\,(h - \ell - c)} + \left( 1 - \left( \frac{p_B - \ell}{1 - \ell} \right) \left( \frac{1 - h}{h - \ell - c} \right) \right) (1 - \rho_b)\,. \tag{19}$$

Note that this condition prevents $\rho_b$ from getting too close to 1 for large $\lambda$, because $p_B \to h$ as $\lambda \to 1/2$ so that the first term on the RHS of (19) is negative. As a result, we will identify below an upper bound $\overline{\lambda}$ above which communication cannot be beneficial. This is the barrier to beneficial communication stemming from the extra condition (ICh$_{BB}$).

From Proposition 2 ($ii'$), the most efficient equilibrium is achieved at $\bar{\mathbf{x}}$ that binds both constraints (18) and (19). Hence, communication is beneficial if $\bar{\rho}_b = (1-\lambda)\bar{x}_{Bb} + \lambda \bar{x}_{Gb} > \rho^*_b$ at this solution. To facilitate comparison, we rearrange $\rho^*_b$ in (6) as

$$1 - \rho^*_b \; = \; \frac{c}{(h - \ell)^2 - c(1 - \ell)} \left( \frac{1 - \delta}{\delta} \right) \; = \; \frac{c}{c^* - c} \times \frac{1 - \delta}{\delta(1 - \ell)} \quad \text{for } c < c^*.$$

Similarly, we solve for $\bar{x}_{Bb}$ and $\bar{x}_{Gb}$ from (18) and (19) and express $\bar{\rho}_b$ as

$$1 - \bar{\rho}_b \; = \; \frac{\Lambda(\lambda) + c}{\Lambda(\lambda)\frac{h-1}{1-\ell} + c^* - c} \times \frac{1 - \delta}{\delta(1 - \ell)} \quad \text{for } \; \lambda > \widetilde{\lambda} \; \text{ and } \; \widehat{c}(\lambda) < c < \overline{c}(\lambda)$$

where

$$\Lambda(\lambda) \; := \; \lambda \left( \frac{p_G - p_B}{1 - p_B} \right) (1 - \ell) - h + p_B \; = \; \frac{c^* - \overline{c}(\lambda)}{1 - h}. \tag{20}$$

[16] In fact, $c \leq \widehat{c}(\lambda)$ is precisely the condition that this configuration satisfies (ICh$_{BB}$).

Note that $1 - \bar{\rho}_b$ is an increasing function of $\Lambda(\lambda)$ and is equal to $1 - \rho_b^*$ when $\Lambda(\lambda) = 0$. Thus, whether $\bar{\rho}_b$ exceeds $\rho_b^*$ or not is independent of $c$ in the current case. Specifically, for $\lambda > \widetilde{\lambda}$ and $c \in (0, c^*) \cap (\widehat{c}(\lambda), \overline{c}(\lambda))$, we have

$$\bar{\rho}_b > \rho_b^* \iff \Lambda(\lambda) < 0 \iff c^* < \overline{c}(\lambda) \iff \underline{c}(\lambda) < \widehat{c}(\lambda) \tag{21}$$

where the last equivalence follows from $\overline{c}(\lambda) - c^* = (1-h)(\widehat{c}(\lambda) - \underline{c}(\lambda))$. It is verified straightforwardly that $\overline{c}(\lambda)$ decreases in $\lambda > \widetilde{\lambda}$ and hits $c^*$ at

$$\lambda = \overline{\lambda} := \frac{(1-h)(3h-\ell)}{2(2h-1)(h-\ell)} \left( \sqrt{1 + \frac{4(2h-1)(h-\ell)h}{(3h-\ell)^2(1-h)}} - 1 \right) > \widetilde{\lambda}.$$[17]

Consequently, together with an earlier assertion that $\widehat{c}(\lambda) < \overline{c}(\lambda)$ for $\lambda > \widetilde{\lambda}$, we deduce that

$$\begin{cases} \lambda \in (\widetilde{\lambda}, \overline{\lambda}) & \Longrightarrow (\widehat{c}(\lambda), c^*) \subset (\underline{c}(\lambda), \overline{c}(\lambda)) \textit{ and } \bar{\rho}_b > \rho_b^* \text{ on } c \in (\widehat{c}(\lambda), c^*) \\ \lambda \geq \overline{\lambda} & \Longrightarrow (\underline{c}(\lambda), \overline{c}(\lambda)) \subset (\widehat{c}(\lambda), c^*) \textit{ and } \bar{\rho}_b \leq \rho_b^* \text{ on } c \in (\widehat{c}(\lambda), \overline{c}(\lambda)). \end{cases} \tag{22}$$

We now combine (22) with the condition (16) for beneficial communication when (17) holds. First, (16) implies communication is beneficial when $\underline{c}(\lambda) < c < \widehat{c}(\lambda)$ if $\lambda \in (\widetilde{\lambda}, \overline{\lambda})$, but is vacuous if $\lambda \geq \overline{\lambda}$ because $\widehat{c}(\lambda) < \underline{c}(\lambda)$ by (21). For $\lambda < \widetilde{\lambda}$, (16) implies beneficial communication for $c \in (\underline{c}(\lambda), \overline{c}(\lambda)) \cap (0, c^*)$. Finally, when $c \geq c^*$, communication is beneficial whenever a FSGV exists, i.e., $c \leq \overline{c}(\lambda)$. Since $\overline{c}(\lambda) \leq c^*$ implies $c^* \leq \underline{c}(\lambda)$ as will be shown shortly, we establish the following result on the potential for communication to improve efficiency:

**Proposition 4** *$\overline{v}_F > v^*$ for large enough $\delta$ if and only if*

$$\lambda < \overline{\lambda} \textit{ and } \underline{c}(\lambda) < c < \overline{c}(\lambda).$$

*Proof.* In the Appendix. ■

Therefore, communication does not help if the signal is too unreliable ($\lambda \geq \overline{\lambda}$) or effort cost is either too small or too large. Finally, we elaborate on these boundaries at which communication ceases to be beneficial.

To get more insight about the upper bound, $\overline{\lambda}$, of the observation error that allows beneficial communication, we derive the following comparative statics on $\overline{\lambda}$ rewritten as a function of $\beta = \ell/h$ and $h$:

$$\overline{\lambda}(\beta, h) = \frac{(1-h)(3-\beta)}{2(2h-1)(1-\beta)} \left( \sqrt{1 + \frac{4(2h-1)(1-\beta)}{(3-\beta)^2(1-h)}} - 1 \right).$$

[17] If $h = \frac{1}{2}$, then $\overline{\lambda} = \frac{1}{3-2\ell} > \widetilde{\lambda}$.

**Lemma 2** *$\overline{\lambda}$ increases in $\beta = \ell/h$ with $\lim_{\beta\to 1} \overline{\lambda} = 1/2$, and decreases in $h$.*

*Proof.* In the Appendix. ■

Recall that increasing $\ell$ (equivalently, increasing $\beta$ given $h$) or decreasing $h$ aggravates the moral hazard problem owing to the worsening *ex-post* monitoring of effort by the delivered quality. Decreasing $h$ given $\beta$ has the same effect. Such changes in the environment permit less precise communication to enhance efficiency, pushing up the upper bound $\overline{\lambda}$. In Figure 2 we show how this upper bound changes with $h$ and $\beta$. Roughly, $\overline{\lambda}$ stays above 0.3 for $h < 0.7$ even as $\beta \to 0$ where $\overline{\lambda}$ is lowest.

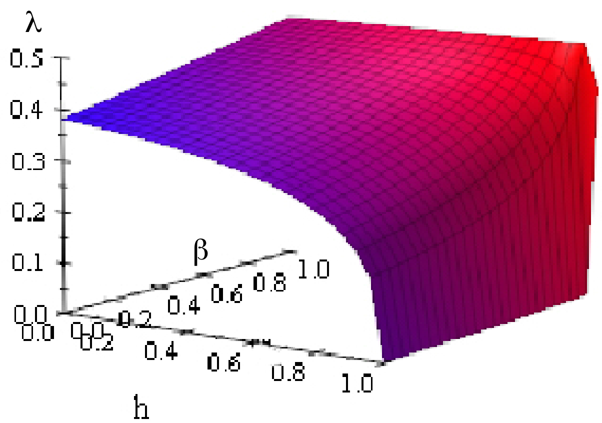



**Figure 2:** $\overline{\lambda}(\beta, h)$

For a given value $\lambda < \overline{\lambda}$, the lower bound of effort cost $c$ for beneficial communication is $\underline{c}(\lambda) > 0$. Note that $\underline{c}(0) = \underline{c}(0.5) = 0$ and $\underline{c}(\lambda)$ is single-peaked at $\lambda = \breve{\lambda}$, the point at which $\overline{v}_F$ in (15) bottoms out. Once again, the non-monotonicity stems from the two countervailing effects of more precise signal on truth-telling incentives mentioned earlier.

The upper bound of $c$ for beneficial communication, $\overline{c}(\lambda)$, is also the upper bound for existence of a FSGV. We have shown above that $(\underline{c}(\lambda), \overline{c}(\lambda))$ is nonempty for $\lambda \in [\widetilde{\lambda}, \overline{\lambda})$ and converges to the full range $(0, h - \ell)$ as $\lambda \to 0$. However, if $\ell$ is sufficiently close to $h$, it may happen that $(\underline{c}(\lambda), \overline{c}(\lambda)) = \emptyset$ for some intermediate values of $\lambda \in (0, \widetilde{\lambda})$ because $\overline{c}(\lambda)$ dips below $c^*$ and $\underline{c}(\lambda)$ pushes above $c^*$ at the same time. We noted above that more severe moral hazard allows less precise signal to be beneficial; on the other hand, it shrinks the range of $c$ for beneficial communication for a given level precision.

To see when $(\underline{c}(\lambda), \overline{c}(\lambda))$ is empty, recall that $\underline{c}(\lambda)$ is the lower bound of $c$ such that the configuration $\hat{\mathbf{x}}$ that binds (ICB) would support a FSGV $v_F(\hat{\mathbf{x}})$ higher than $v^*$ provided that (IC$\text{h}_{GB}$) is also satisfied. This is indeed satisfied if $\underline{c}(\lambda) < c^*$ because (IC$\text{h}_{GB}$) is weaker than (ICh), the corresponding condition without communication, which is satisfied for any $c < c^*$. By the same token, therefore, if $\underline{c}(\lambda) < c^*$ then a FSGV higher than $v^*$ is supported for all $c \in (\underline{c}(\lambda), c^*)$, thus implying that $c^* < \overline{c}(\lambda)$. Therefore, $\overline{c}(\lambda) \leq c^*$ may happen only if $c^* \leq \underline{c}(\lambda)$, which is precisely the case stated below.

**Corollary 1** *Communication cannot improve efficiency regardless of $c$ for large enough $\delta$ if and only if $\overline{c}(\lambda) \leq c^*$. This inequality holds for $(i)$ all $\lambda \in [\overline{\lambda}, 1/2)$ and $(ii)$ $\lambda$ in a non-empty interval in the interior of $[0, \widetilde{\lambda}]$ only if $\ell$ is close enough to $h$.*

*Proof.* We establish that when $\lambda \leq \widetilde{\lambda}$, the condition $\overline{c}(\lambda) \leq c^*$ is equivalent to $\underline{c}(\lambda) \geq c^*$ in the proof of Proposition 4. Together with (22) and Proposition 4, this proves the Corollary except part $(ii)$.

For $\widetilde{\lambda} \leq \lambda < \overline{\lambda}$, we have $\overline{c}(\lambda) > c^*$ by (22) so that communication improves efficiency for some $c$. For $\lambda < \widetilde{\lambda}$, since $\underline{c}(\lambda)$ is single-peaked at $\breve{\lambda}$, the condition $\overline{c}(\lambda) \leq c^* \Leftrightarrow \underline{c}(\lambda) \geq c^* = \frac{(h-\ell)^2}{1-\ell}$ holds in a non-empty interval of $\lambda$ if

$$\frac{h-\ell}{1-\ell} \leq \max_{\lambda \leq \widetilde{\lambda}} \frac{\lambda\,(1-2\lambda)}{(1-\lambda)\,(\lambda + h\,(1-2\lambda))} \tag{23}$$

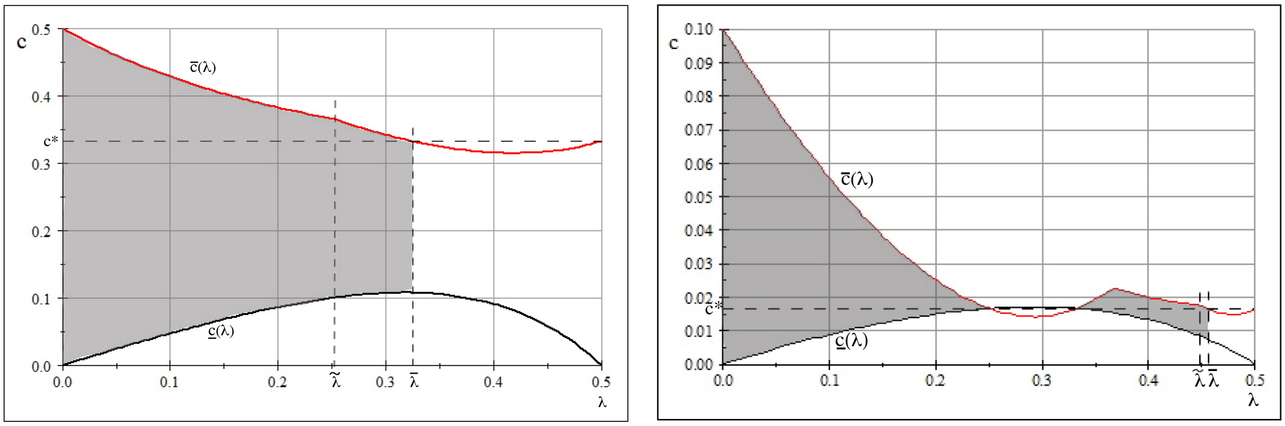
but for no $\lambda$ otherwise. The LHS of (23) decreases from $h$ to 0 as $\ell$ increases from 0 to $h$, while the RHS increases in $\ell$ (strictly for low $\ell$) from 0 at $\ell = 0$ because $\widetilde{\lambda}$ increases in $\ell$ from 0 at $\ell = 0$. Hence, the inequality holds when $\ell$ is large enough and for a non-empty interval in the interior of $[0, \widetilde{\lambda}]$ because $\underline{c}(\lambda)$ is single-peaked, $\underline{c}(0) < c^*$ and $\underline{c}(\widetilde{\lambda}) < c^*$. ■

Therefore, communication can be beneficial for the entire range of effort cost $c$ when $\lambda \to 0$; the range of $c$ gradually shrinks as $\lambda$ increases from 0 but may expand temporarily as $\lambda$ increases further before it reaches the upper bound $\overline{\lambda}$ at which the range ceases to exist. In addition, the range may disappear for some intermediate values of $\lambda < \widetilde{\lambda}$ and reappear for higher values of $\lambda$. Such non-monotonic changes reflect the fundamental trade-off we highlighted, namely, that noisier signals reduce not only the price differential, which is the short-run gain from overstating a bad signal, but also the long-run loss by reducing the risk of getting detected.

This observation leads to a counter-intuitive measure to address moral hazard in sellers: if the seller's observation of quality is not very precise and cannot be improved easily, then making it noisier may facilitate truthful communication by rendering overstatement less attractive and thereby, enhance efficiency.

In line with the discussions in this section, Figure 3-(a) illustrates in grey the parameter values $(c, \lambda)$ for which pre-trade communication is beneficial when $\ell$ is not too close to $h$ with the case $(h, \ell) = (0.75, 0.25)$: it is below the red curve $\overline{c}(\lambda)$ and above the black curve $\underline{c}(\lambda)$ for $\lambda$ lower than $\overline{\lambda}$ at which $\overline{c}(\lambda)$ intersects the horizontal dashed line at the level $c^*$.

Figure 3-(b) illustrates the corresponding area when $\ell$ is closer to $h$ with $(h, \ell) = (0.5, 0.4)$. Notice an intermediate range of $\lambda$ where $\overline{c}(\lambda) \leq c^* \leq \underline{c}(\lambda)$, for which communication cannot be beneficial for any level of the effort cost $c$. Consequently, there are two disconnected regions of parameter values on which communication improves efficiency.




(a) Large gap $h - \ell$ (b) Small gap $h - \ell$

**Figure 3**

# 5 Conclusion

In this paper we have examined how and when cheap-talk communication by a seller can help discipline herself and thereby, enhance efficiency and her equilibrium payoff. Generally speaking, this is possible when the moral hazard problem is neither too mild nor too acute and the seller's information is not too noisy. In this case, the incentives for effort and for truthful communication are interwoven within the same reputation mechanism that determines continuation equilibria based on the seller's past performance, in such a way that truthful communication complements the conventional reputational incentives by permitting immediate reward for effort via more accurate prices. This raises the value of reputation for being trustworthy, which in turn provides credibility to seller's communication.

We have developed our analysis in a setting of experience good sellers, but the insight should be more broadly applicable to situations that involve *interim* communication by actors who are subject to moral hazard and reputation. For instance, managers report about likely performance, academic scholars communicate about intermediate research findings, and doctors update the progress of treatments. As already mentioned, communication could be used to move forward some reward in relational contracts.

In our model where actions, outcomes and signals are binary, the maximum equilibrium value is characterized by a well-defined linear program. This allowed us to study systematically the impacts of signal precision on the incentives for truthful communication and consequently, to characterize fully the optimal communication and effort strategy as well as the environments in which communication enhances welfare. It is an interesting agenda, which we leave for future work, to investigate whether our findings are driven by the discrete nature of our model, or they extend to richer environments that accommodate more flexible communication and actions.[18]

# Appendix

## A Proof of Proposition 1

In the main text, we have identified the maximum equilibrium value $v^*$ in (6), presuming it exists. To verify that it indeed exists, consider a sequence of equilibrium values $\{v^n\}$ converging to the supremum of equilibrium values, $v^* > \underline{v}$. Then, for every $n$, the continuation values in the second period can be expressed as a convex combination of $v^*$ and $\underline{v}$. Hence, taking a subsequence if necessary, by continuity, $\lim_{n\to\infty} e_1^n$ is optimal given the limit convex combination of $v^*$ and $\underline{v}$ as the continuation values, generating $v^*$ as a self-generated value and thus, as an equilibrium value.

Thus, we have established that $v^*$ in (6) is the maximum equilibrium value if $c < c^*$ and $\delta^*(c) < \delta < 1$; and that $\underline{v}$ is the maximum equilibrium value otherwise. Then, $\underline{v}$ is clearly the unique equilibrium value if $c \geq c^*$.

It remains to verify that every $v \in (\underline{v}, v^*)$ is an equilibrium value if $c < c^*$. It has been shown that $v^*$ in (6) is the maximum self-generated value, obtained with $p_1 = h$, $\rho_g = 1$, and $\rho_b = \rho_b^*$ which is the largest value compatible with (ICh). Recall $v(p_1, \rho_g, \rho_b)$ from (5).

As $(\rho_g - \rho_b)(v(p_1, \rho_g, \rho_b) - \underline{v})$ decreases in $\rho_b$, the value $v(h, 1, \rho_b)$ constitutes a self-generated value for all $\rho_b \in [0, \rho_b^*]$ as (ICh) is satisfied. Thus, any value between $v(h, 1, 0)$ and $v(h, 1, \rho_b^*)$ is self-generated. Analogously, fixing $p_1 = h$ and $\rho_g < 1$, any value between $v(h, \rho_g, 0)$ and $v(h, \rho_g, \widehat{\rho}_b(\rho_g))$ is self-generated, where $\widehat{\rho}_b(\rho_g)$ is the largest non-negative value compatible with (ICh) given $p_1 = h$ and $\rho_g$, when it exists. Note that $\widehat{\rho}_b(\rho_g)$ increases in $\rho_g$ because $(\rho_g - \rho_b)(v(p_1, \rho_g, \rho_b) - \underline{v})$ increases in $\rho_g$. Thus, there is unique $\underline{\rho_g}(h) \in (0, 1)$ such that $\widehat{\rho}_b(\underline{\rho_g}(h)) = 0$. By continuity, the set of self-generated values associated with $p_1 = h$ is the interval $[v(h, \underline{\rho_g}(h), 0), v(h, 1, \rho_b^*)]$.

Similarly, for each $p_1 \in (\ell, h)$ the set of self-generated values is a closed interval[19] with the minimum value of $v(p_1, \underline{\rho_g}(p_1), 0)$ so long as $\underline{\rho_g}(p_1) \leq 1$. As $\underline{\rho_g}(p_1)$ solves (ICh) as equality when $\rho_b = 0$, i.e., $\underline{\rho_g}(p_1) = \frac{c}{\delta(c\ell + (h-\ell)(p_1 - \ell))}$, we have $\underline{\rho_g}(p_1) = 1$ when $p_1 = \frac{c(1-\delta\ell)}{\delta(h-\ell)} + \ell$ for which $v(p_1, 1, 0) = \widetilde{v} := \frac{(1-\delta)c}{\delta(h-\ell)} + \underline{v}$. Therefore, the set of all self-generated values is the

[18]If the seller's signal is perfect, however, our logic extends straightforwardly to the case of continua of effort/quality levels and the full efficiency is achieved.

[19]To be precise, it is $\{v(p_1, \rho_g, \rho_b) \,|\, \text{(ICh) binds for some } (\rho_g, \rho_b) \in [0, 1]^2\}$.

interval $[\widetilde{v}, v^*]$.

Observe from $(1-\delta)c = \delta(h-\ell)(\widetilde{v} - \underline{v})$ that the seller is indifferent between exerting $h$ and $\ell$ if the continuation payoff in period two is $\widetilde{v}$ if $q_1 = g$ and is $\underline{v}$ if $q_1 = b$. Given such continuation payoffs, it is optimal for the seller to exert $h$ with any probability $e_1 \in [0,1]$, generating the seller's value of $(1-\delta)p_1 + \delta(e_1 h + (1-e_1)\ell)(\widetilde{v} - \underline{v}) + \delta\underline{v}$ where $p_1 = e_1(h-\ell)+\ell$. These values range from $v_{(1)} = (1-\delta)\ell + \delta(\ell(\widetilde{v}-\underline{v})+\underline{v}) = \underline{v}+\delta\ell(\widetilde{v}-\underline{v}) < \widetilde{v}$ to $(1-\delta)h + \delta(h(\widetilde{v}-\underline{v})+\underline{v}) > \widetilde{v}$. We refer to the values in the interval $[v_{(1)}, \widetilde{v}]$ as self-generated with one lag. These are clearly equilibrium values.

Next, consider the seller's value from exerting $\ell$ for the first $t \geq 1$ periods, after which the continuation value is $v_g \in [v_{(1)}, \widetilde{v}]$ if all qualities have been good up to then but is $\underline{v}$ otherwise. As the prices are $\ell$ in the first $t$ periods, the set of seller's values obtained as such is

$$\{\underline{v} + \delta^t \ell^t (v_g - \underline{v}) \,|\, v_{(1)} \leq v_g \leq \widetilde{v}]\} \;=\; [\underline{v} + \delta^{t+1}\ell^{t+1}(\widetilde{v} - \underline{v}), \underline{v} + \delta^t \ell^t (\widetilde{v} - \underline{v})]$$

which we call self-generated values with $t+1$ lags. For each such value $v$, the initial effort choice of $\ell$ is optimal given that the continuation value is $\underline{v}$ if $q_1 = b$ and is a self-generated value $v'(< \tilde{v})$ with $t$ lags if $q_1 = g$ such that $\underline{v} + \delta\ell(v' - \underline{v}) = v$. As such, every value $v \in (\underline{v}, v_{(1)})$ is self-generated with $t$ lags for some $t \geq 2$, which clearly is an equilibrium value.

Lastly, we show that every value $v \in [\underline{v}, v^*]$ is an equilibrium value even without public randomization. As shown above, any such value $v$ is either self-generated or self-generated with lags, with an associated first period strategy $e_1$ and a continuation value $v_q \in [\underline{v}, v^*]$ conditional on the first period quality $q \in \{g, b\}$. As each $v_q$ is self-generated or self-generated with lags, conditional on the first period quality $q \in \{g, b\}$, the associated second period strategy and subsequent continuation values are specified accordingly. Proceeding recursively, one can determine a strategy $e$ by specifying $e(H_t)$ for every possible history. By construction, $e(H_t)$ is optimal conditional on the history relative to the price schedule defined by $e$ and $v$ is the associated value, completing the proof. ∎

# B Proof of Lemma 1

A FSGV is supported by a configuration $\mathbf{x} \in [0,1]^4$ that satisfies (ICB), (ICG), (ICh$_{GB}$) and (ICh$_{BB}$). Since (ICB) and (ICG) imply (D), it follows that the solution value to $(\overline{P})$, denoted by $\bar{\bar{v}}_F$, is no lower than the maximum FSGV $\overline{v}_F$ presuming that it exists.

Consider a solution to $(\overline{P})$ which is a configuration $\mathbf{x}$ that satisfies (ICh$_{GB}$), (ICh$_{BB}$), (ICB) and (D), so that $\Delta_b = x_{Bb} - x_{Gb} > 0$. If both (ICB) and (D) are slack at $\mathbf{x}$, one can reduces $x_{Bb}$ while increasing $x_{Gb}$ to keep $\rho_b = (1-\lambda)x_{Bb} + \lambda x_{Gb}$ constant until either (ICB) or (D) binds. Since this keeps $v_F(\mathbf{x})$ and thus (ICh$_{GB}$) intact while loosening (ICh$_{BB}$), the modified configuration also supports $\bar{\bar{v}}_F$. Hence, we may assume that (D) or (ICB) binds at the solution $\mathbf{x}$ to $(\overline{P})$.

If only (D) binds at $\mathbf{x}$, then $x_{Bq} > 0$ and $x_{Gq} < 1$. If $x_{Bg} < x_{Bb}$, raise $x_{Bg}$ and $x_{Gg}$ by the same amount while reducing $x_{Bb}$ and $x_{Gb}$ by the same amount to keep $h\rho_g + (1-h)\rho_b$ intact. Since this keeps (ICB) and (D) intact and loosens (ICh$_{GB}$) and (ICh$_{BB}$), we may assume that $x_{Bg} \geq x_{Bb}$ at the solution $\mathbf{x}$ to $(\overline{P})$ that binds (D) but not (ICB). Then, $x_{Gg}$ and $x_{Gb}$ may be raised by the same amount until (ICB) binds, which clearly keeps (D) intact and loosens (ICh$_{BB}$); it also loosens (ICh$_{GB}$) because the direction of change in $(\rho_g - \rho_b)(v_F - \underline{v})$ is captured by

$$\frac{\partial}{\partial x_{Gg}}\left(\frac{\rho_g - \rho_b}{1-\delta(h\rho_g + (1-h)\rho_b)}\Big|_{x_{Gb}=x_{Bb}-\Delta_g}\right) = \frac{1-\delta x_{Bb} + \lambda(\delta(x_{Bg}+x_{Bb})-2)}{(1-\delta(h\rho_g+(1-h)\rho_b))^2}$$

which is positive since it is linear in $\lambda$ and positive at both $\lambda = 0$ and $1/2$.

Consequently, there is a solution $\mathbf{x}$ to $(\overline{P})$ that binds (ICB). Then, (D) implies that (ICG) holds at $\mathbf{x}$, further implying that $\mathbf{x}$ supports a FSGV which is at most $\overline{v}_F$. Since $\overline{v}_F \leq \bar{\bar{v}}_F$ as asserted earlier, we have established the equivalence of the maximum FSGV $\overline{v}_F$ and the solution value to $(\overline{P})$.

Finally, at a solution $\mathbf{x}$ to $(\overline{P})$ that binds (ICB), the seller is indifferent between announcing $G$ and $B$ when her posterior is $\pi_{\mathbb{b}}$. Hence, upon observing $s = \mathbb{g}$, an $\ell$-seller would find it uniquely optimal to announce $G$ (resp. $B$) if $\pi_{\mathbb{b}} < \pi'_{\mathbb{g}} \Leftrightarrow \lambda < \widetilde{\lambda}$ (resp. $\pi_{\mathbb{b}} > \pi'_{\mathbb{g}} \Leftrightarrow \lambda > \widetilde{\lambda}$) by (8). This implies that (ICh$_{BB}$) is slack at $\mathbf{x}$ if $\lambda < \widetilde{\lambda}$ while (ICh$_{GB}$) is slack if $\lambda > \widetilde{\lambda}$.

# C Proof of Proposition 2

Since $\Delta_q$ and $v_F(\mathbf{x})$ increase in $x_{Bq}$ for each $q \in \{g, b\}$, so does the LHS of (ICB). Moreover, with $x_{Bg} = x_{Bb} = 1$, the LHS of (ICB) increases in $x_{Gg}$ and decreases in $x_{Gb}$ for large enough $\delta$ because the respective derivative of the LHS divided by $(1-\delta)$ converges, as $\delta \to 1$, to

$$\frac{h(1-h)(h-\ell-c)(1-x_{Gb})(1-2\lambda)}{(1-\lambda-h(1-2\lambda))X^2} > 0 \quad \text{and} \quad \frac{h(1-h)(h-\ell-c)(x_{Gg}-1)(1-2\lambda)}{(1-\lambda-h(1-2\lambda))X^2} < 0$$

where $X = \lambda(1-x_{Gb}) + h(1-(1-\lambda)x_{Gg} - (2-x_{Gb})\lambda)$. Hence, for (ICB) to be satisfied by some configuration, it must be satisfied by $\mathbf{x} = (x_{Bg}, x_{Gg}, x_{Bb}, x_{Gb}) = (1,1,1,0)$, which is the case if and only if

$$c \leq h - \ell - \frac{p_G - p_B}{1-p_B}\left(\frac{1}{\delta} - h - (1-h)(1-\lambda)\right) \longrightarrow c_{ICB} := h - \ell - \lambda(1-h)\frac{p_G - p_B}{1-p_B}$$

as $\delta \to 1$ where the convergence is from below. Therefore,

[C1] (ICB) holds for some configuration $\mathbf{x}$ for large enough $\delta$ if and only if $c < c_{ICB}$, and in this case it holds at $\mathbf{x} = (1, 1, 1, 0)$.

Moreover, in this case the value $v_F(\mathbf{x})$ is maximized subject to (ICB) at $\hat{\mathbf{x}} = (1, 1, 1, \hat{x}_{Gb})$ where $\hat{x}_{Gb} < 1$ is the unique value at which (ICB) binds. We further establish the following.

[C2] If $\overline{v}_F$ is supported by a configuration $\mathbf{x}$ that leaves ($\text{ICh}_{GB}$) and ($\text{ICh}_{BB}$) slack and $x_{Gb} > 0$, then $\mathbf{x} = \hat{\mathbf{x}}$.

To prove this, consider a configuration $\mathbf{x} = (x_{Bg}, x_{Gg}, x_{Bb}, x_{Gb})$ that supports $\overline{v}_F$ as such. We now prove [C2] in four steps.

Step 1. $x_{Bg} = 1$: If $x_{Bg} < 1$, increase $x_{Bg}$ and $x_{Gg}$ slightly keeping $\Delta_g$ intact, which would increase $v_F(\mathbf{x})$ without violating any constraint of the program ($\overline{P}$). As this would contradict $\mathbf{x}$ supporting $\overline{v}_F$, we deduce that $x_{Bg} = 1$ must hold.

Step 2. $x_{Gg} = 1$ or $x_{Bb} = 1$: If $x_{Gg} < 1$ and $x_{Bb} < 1$, one can increase $x_{Gg}$ and $x_{Bb}$ by the same amount, i.e., $dx_{Gg} = dx_{Bb} = dx > 0$. If $\pi_{\mathbb{b}} < 1 - \pi_{\mathbb{b}}$, this raises $v_F(\mathbf{x})$ while relaxing (ICB) and (D), a contradiction to $\mathbf{x}$ supporting $\overline{v}_F$. If $\pi_{\mathbb{b}} > 1 - \pi_{\mathbb{b}}$, in addition to increasing $x_{Gg}$ and $x_{Bb}$ as above, one may reduce $x_{Gb}$ so that $(\pi_{\mathbb{b}}\Delta_g + (1 - \pi_{\mathbb{b}})\Delta_b)$ is constant, i.e., $(1 - 2\pi_{\mathbb{b}})dx = (1 - \pi_{\mathbb{b}})dx_{Gb}$. This increases $v_F(\mathbf{x})$ because

$$d(h\rho_g + (1-h)\rho_b) = \left(1 - \lambda + (1-h)\lambda\frac{1 - 2\pi_{\mathbb{b}}}{1 - \pi_{\mathbb{b}}}\right) dx > (1 - 2\lambda h)\, dx$$

where the inequality follows from $\frac{1-2h}{1-h} < \frac{1-2\pi_{\mathbb{b}}}{1-\pi_{\mathbb{b}}} < 0$, thus relaxing (ICB) as well as (D), again a contradiction.

Step 3. $x_{Gg} = x_{Bb} = 1$: If $x_{Gg} = 1 > x_{Bb}$, increase $x_{Bb}$. If $x_{Gg} < 1 = x_{Bb}$, increase $x_{Gg}$ and decrease $x_{Gb}$ in such a way that $\pi_{\mathbb{b}}\Delta_g + (1-\pi_{\mathbb{b}})\Delta_b$ is intact, i.e., $\pi_{\mathbb{b}}dx_{Gg} + (1-\pi_{\mathbb{b}})dx_{Gb} = 0$ and thus

$$d(h\rho_g + (1-h)\rho_b) = h(1-\lambda)dx_{Gg} + (1-h)\lambda dx_{Gb} > (1-\lambda)[hdx_{Gg} + (1-h)dx_{Gb}] > 0.$$

Either case, $v_F(\mathbf{x})$ increases while maintaining (ICB) and (D), a contradiction to $\mathbf{x}$ supporting $\overline{v}_F$.

Step 4. $\hat{\mathbf{x}}$ supports $\overline{v}_F$: By Steps 1–3, $\overline{v}_F$ is supported by a configuration $\mathbf{x} = (1, 1, 1, x_{Gb})$ at which ($\text{ICh}_{GB}$) and ($\text{ICh}_{BB}$) are slack, as well as (D). Therefore, (ICB) must bind because otherwise $x_{Gb}$ may be increased without violating any constraint. This proves [C2].

When [C2] applies, the solution is $\hat{\mathbf{x}}$ where

$$\hat{x}_{Gb} = \frac{\delta(1 - p_B)\,(h - \ell - c) - (p_G - p_B)(1 - \delta + \delta(1-h)\lambda)}{\delta[(1 - p_B)\,(h - \ell - c) - (p_G - p_B)\,(1 - h)\,\lambda]} \tag{24}$$

and $\overline{v}_F(\hat{\mathbf{x}})$ is routinely calculated as the formula in (15).

The proof now proceeds differently between the two cases $\lambda \leq \widetilde{\lambda}$ and $\lambda > \widetilde{\lambda}$.

## C.1 Case where $\lambda \leq \widetilde{\lambda}$

In this case we focus on ($\text{ICh}_{GB}$), (ICB) and (D) because the three conditions, with (ICB) binding, imply ($\text{ICh}_{BB}$). As $v_F(\mathbf{x})$ increases in $\rho_g$, for ($\text{ICh}_{GB}$) to hold at any $\mathbf{x}$ it must holds at $\rho_g = 1$ which is written as

$$\frac{\delta\,(1-\rho_b)\,(1-h)}{(1-\delta h-\delta\,(1-h)\,\rho_b)} \geq \frac{(c-(h-\ell)\,(1-2\lambda)\,(p_G-p_B))\,(1-h)}{(h-\ell)\,(h-\ell-c)}. \tag{25}$$

The LHS decreases in $\rho_b$, and it is less than 1 and converges to 1 as $\delta \to 1$ for all $\rho_b < 1$. Hence, ($\text{ICh}_{GB}$) may hold at some $\mathbf{x}$ for large enough $\delta$ if and only if the RHS is strictly less than 1, which is calculated to be the case if and only if

$$c \;<\; c_{ICh} \;:= c^* + (1-2\lambda)\frac{(h-\ell)(1-h)}{1-\ell}(p_G-p_B).$$

Note that, in this case, ($\text{ICh}_{GB}$) holds at every $\mathbf{x} = (1,1,x_{Bb},x_{Gb}) \neq (1,1,1,1)$ for $\delta$ large enough because the LHS of (25) converges to 1 as $\delta \to 1$. Together with [C1], therefore, both ($\text{ICh}_{GB}$) and (ICB) hold at some $\mathbf{x}$ when $\delta$ is large enough if and only if $c < \min\{c_{ICh}, c_{ICB}\} = \overline{c}(\lambda)$. Since in this case they both hold at $\mathbf{x} = (1,1,1,0)$ which also satisfies (D), a FSGV exists for large enough $\delta$ if and only if $c \in (0, \overline{c}(\lambda))$, thus so does the maximum FSGV, $\overline{v}_F$, by the Maximum theorem (as the objective function of ($\overline{P}$) is continuous subject to a compact constraint set).

Suppose $c < \overline{c}(\lambda)$, so that $\overline{v}_F$ exists. Since $\mathbf{x} = (1,1,1,0)$ satisfies ($\text{ICh}_{GB}$), (ICB) and (D) strictly in this case, we have $\overline{v}_F > v_F(1,1,1,0)$. Thus, any configuration $\mathbf{x} = (x_{Bg}, x_{Gg}, x_{Bb}, x_{Gb})$ supporting $\overline{v}_F$ must have $x_{Gb} > 0$.

Hence, if $\overline{v}_F$ is supported by a configuration at which ($\text{ICh}_{GB}$) is slack, it must be $\hat{\mathbf{x}} = (1,1,1,\hat{x}_{Gb})$ by [C2], and thus, ($\text{ICh}_{GB}$) must be slack at $\hat{\mathbf{x}}$. Given $x_{Gg} = x_{Bg} = x_{Bb} = 1$, it is routinely verified that the LHS of ($\text{ICh}_{GB}$) decreases in $x_{Gb}$, hence it binds at a unique $x'_{Gb} < 1$ and hold at all lower $x_{Gb}$. It is straightforward to verify (by Mathematica) that $\hat{x}_{Gb} \leq x'_{Gb}$ for large enough $\delta$ if and only if

$$c \leq (h-\ell)(p_G-p_B)\Big(1-2\lambda+\frac{\lambda}{1-p_B}\Big) = \widehat{c}(\lambda).$$

This verifies that $\overline{v}_F$ is supported by $\mathbf{x} = (1,1,1,\hat{x}_{Gb})$ for large enough $\delta$ if $c \leq \widehat{c}(\lambda)$, establishing Proposition 2 for $\lambda \leq \widetilde{\lambda}$ when $c \leq \widehat{c}(\lambda)$.

If $c > \widehat{c}(\lambda)$, on the other hand, $\overline{v}_F$ cannot be supported by a configuration at which ($\text{ICh}_{GB}$) is slack by [C2], thus it is supported by $\mathbf{x} = (x_{Bg}, x_{Gg}, x_{Bb}, x_{Gb})$ that binds ($\text{ICh}_{GB}$). Note that $\mathbf{x}' = (1,1,1,x'_{Gb})$ binds ($\text{ICh}_{GB}$) and satisfies (ICB) and (D) loosely. If $v_F(\mathbf{x}) > v_F(\mathbf{x}')$, then $\rho_b$ must be higher at $\mathbf{x}$ than $\mathbf{x}'$, but then the LHS of ($\text{ICh}_{GB}$) is lower at $\mathbf{x}$ than $\mathbf{x}'$ because the LHS of ($\text{ICh}_{GB}$) increases in $\rho_g$ and decreases in $\rho_b$, a contradiction to $\mathbf{x}$ supporting $\overline{v}_F$. Therefore, $\overline{v}_F$ is supported by $\mathbf{x}'$, and $\overline{v}_F(\mathbf{x}')$ is calculated (by Mathematica) as

$$\overline{v}_F(\mathbf{x}') = -\frac{c(1-\ell)}{h-\ell} + \frac{1-h(1-2\lambda)^2-3\lambda(1-\lambda)}{h(1-2\lambda)^2(1-h)+\lambda(1-\lambda)}h.$$

However, since (ICB) is slack at $\mathbf{x}'$, there is flexibility in choosing $x_{Bb}$ and $x_{Gb}$ because only $\rho_b$ matters for $v_F(\mathbf{x})$ and ($\text{ICh}_{GB}$). Thus we can also obtain $\overline{v}_F$ with (ICB) binding, by reducing $x_{Bb}$ from 1 and increasing $x_{Gb}$ from $x'_{Gb}$ keeping $\rho_b$ intact until (ICB) binds. Note that (D) holds because $\Delta_g = 0$.

This establishes Proposition 2 for $\lambda \leq \widetilde{\lambda}$.

**Discussion of the thresholds**

In fact, the thresholds are related as follows:

$$c_{ICh} - \widehat{c}(\lambda) \;=\; (h-\ell)A(\lambda) \quad \text{and} \quad c_{ICB} - \widehat{c}(\lambda) \;=\; (1-\ell)A(\lambda)$$

where

$$\begin{aligned} A(\lambda) \;&:=\; (1-\ell)\left[\frac{h-\ell}{1-\ell}(1-(p_G-p_B)(1-2\lambda)) - \frac{\lambda(p_G-p_B)}{1-p_B}\right] \\ &=\; \frac{\lambda h\left[\frac{h-\ell}{h(1-\ell)}(1-\lambda)^2-(1-2\lambda)(h\lambda+(1-h)(1-\lambda))\right]}{(1-\lambda)(h(1-\lambda)+(1-h)\lambda)(h\lambda+(1-h)(1-\lambda))}. \end{aligned} \tag{26}$$

Hence either $\widehat{c}(\lambda) < \overline{c}(\lambda) = c_{ICh} < c_{ICB}$ or $\overline{c}(\lambda) = c_{ICB} < c_{ICh} < \widehat{c}(\lambda)$. The latter case occurs when ($\text{ICh}_{GB}$) is easier to satisfy than (ICB) so that the truth-telling rent, the RHS of (ICB), is sufficient to induce $h$. In the former case ($\text{ICh}_{GB}$) becomes more stringent than (ICB) when the effort cost $c$ is large enough.

Observe that $\widehat{c}(\lambda) < \overline{c}(\lambda)$ when $A(\lambda) > 0$ and $\widehat{c}(\lambda) > \overline{c}(\lambda)$ when $A(\lambda) < 0$. Note that $r := \frac{h-\ell}{h(1-\ell)}$ decreases from 1 to 0 as $\ell$ increases from 0 to $h$. Given any $\lambda$, in particular, $\overline{c}(\lambda) < \widehat{c}(\lambda)$ if $\ell$ is close enough to $h$.

To check the sign of $A(\lambda)$ by that of the term in the bracket of (26), we note that

$$B(\lambda) = r\frac{(1-\lambda)^2}{(1-2\lambda)} - (h\lambda + (1-h)(1-\lambda))$$

is convex and $B(0) = r+h-1$ and $B'(0) < 0$. Since $\widetilde{\lambda} = \frac{r-1+\sqrt{1-r}}{r}$ from $\frac{h(1-r)}{1-rh} = \ell$, we get

$$\begin{aligned} B(\widetilde{\lambda}) \;&=\; \frac{r(1-\sqrt{1-r})^2}{(2-r-2\sqrt{1-r})} - (h(r-1+\sqrt{1-r}) + (1-h)(1-\sqrt{1-r})) \\ &\geq\; \left(\frac{r(1-\sqrt{1-r})}{2-r-2\sqrt{1-r}} - 1\right)(1-\sqrt{1-r}) > 0 \end{aligned}$$

Therefore,

- If $B(0) \le 0 \Leftrightarrow r+h \le 1$, then we have $\widehat{c}(\lambda) < \overline{c}(\lambda) = c_{ICh}$ for $\lambda$ above some threshold while $\overline{c}(\lambda) = c_{ICB} < \widehat{c}(\lambda)$ below the threshold.
- If $r+h > 1$ and $r$ is not too large, then $\widehat{c}(\lambda) < \overline{c}(\lambda) = c_{ICh}$ for $\lambda$ small or close to $\widetilde{\lambda}$, but $\overline{c}(\lambda) = c_{ICB} < \widehat{c}(\lambda)$ holds in some interior interval.
- Finally, when $r+h > 1$ and $r$ is large ($\ell$ is small) so that $\frac{(1-\lambda)^2}{(1-2\lambda)} > 1$, then $\widehat{c}(\lambda) < \overline{c}(\lambda) = c_{ICh}$ for all $\lambda$.

## C.2 Case $\lambda > \widetilde{\lambda}$

In this case we focus on (ICh$_{BB}$), (ICB) and (D) because the three conditions, with (ICB) binding, imply (ICh$_{GB}$). Recall that (ICB) holds for some $\mathbf{x}$ for large enough $\delta$ if and only if $c < c_{ICB}$, in which case $\hat{\mathbf{x}} = (1,1,1,\hat{x}_{Gb})$ maximizes $v_F(\mathbf{x})$ subject to (ICB), and binds (ICB). It is straightforward to verify that (ICh$_{BB}$) holds at $\hat{\mathbf{x}}$ if and only if

$$c \ \le \widehat{c}(\lambda) = h - p_B - \lambda(1-h)\frac{p_G - p_B}{1-p_B} = \frac{(1-2\lambda)^2(1-h)h^2}{(1-\lambda)(h+\lambda(1-2h))(1-h-\lambda(1-2h))}. \quad (27)$$

Since $\widehat{c}(\lambda) < c_{ICB}$ and $\hat{\mathbf{x}}$ satisfies (D), it follows that $\overline{v}_F = v_F(\hat{\mathbf{x}})$ if $c \le \widehat{c}(\lambda)$, establishing Proposition 2 for $\lambda > \widetilde{\lambda}$ and $c \le \widehat{c}(\lambda)$.

Next, for the remaining case that $c > \widehat{c}(\lambda)$ so that (ICh$_{BB}$) fails at $\hat{\mathbf{x}}$, consider a configuration $\mathbf{x} = (x_{Bg}, x_{Gg}, x_{Bb}, x_{Gb})$ that supports $\overline{v}_F$, presuming it exists. First we show that

$$x_{Gg} = 1 \ \text{ or } \ x_{Gb} = 0 \quad (28)$$

must hold. To verify this, suppose to the contrary that $x_{Gg} < 1$ and $x_{Gb} > 0$. Then, one can increase $x_{Gg}$ and decrease $x_{Gb}$, while increasing $h\rho_g + (1-h)\rho_b$, i.e. $\pi_{\mathbb{g}} dx_{Gg} + (1-\pi_{\mathbb{g}})dx_{Gb} > 0$, and increasing $\pi_{\mathbb{b}}\Delta_g + (1-\pi_{\mathbb{b}})\Delta_b$, i.e. $\pi_{\mathbb{b}} dx_{Gg} + (1-\pi_{\mathbb{b}})dx_{Gb} < 0$. This would increase $\overline{v}_F$ while relaxing (ICB), (D) and (ICh$_{BB}$), a contradiction. Hence, (28) must hold.

Also, we verify that

$$\Delta_b = \Delta_g \quad \textit{or} \quad x_{Bg} = 1. \quad (29)$$

Suppose otherwise, i.e., $\Delta_b > \Delta_g$ and $x_{Bg} < 1$. Then, since $\Delta_b > 0$ by (ICB), one can reduce $x_{Bb}$ and increase $x_{Bg}$ while keeping $h\rho_g + (1-h)\rho_b$ constant, so that $h\lambda dx_{Bg} + (1-h)(1-\lambda)dx_{Bb} = 0 \ \Leftrightarrow \ \pi_{\mathbb{b}} dx_{Bg} + (1-\pi_{\mathbb{b}})dx_{Bb} = 0$ and $\ell dx_{Bg} + (1-\ell)dx_{Bb} < 0$ because $\ell < \pi_{\mathbb{b}}$ for $\lambda > \widetilde{\lambda}$. This would keep $v_F(\mathbf{x})$ constant while relaxing (ICh$_{BB}$) and (ICB) because $dx_{Bg} = d\Delta_g$ and $dx_{Bb} = d\Delta_b$. As (D) remains slack, $v_F(\mathbf{x})$ can be increased above $\overline{v}_F$, a contradiction. Thus, (29) must hold.

Given (28) and (29), there are three possibilities in which $\mathbf{x}$ may support $\overline{v}_F$: $(i)$ $x_{Bg} = x_{Gg} = 1$, $(ii)$ $x_{Gb} = 0$ and $x_{Bg} = 1$, $(iii)$ $x_{Gb} = 0$, $\Delta_b = \Delta_g$ and $x_{Bg} < 1$. We examine these possibilities below.

We start with possibility $(i)$ $x_{Bg} = x_{Gg} = 1$, so that $\rho_g = 1$. Solve binding ($\mathrm{ICh}_{BB}$) and (ICB) simultaneously to get the solution:

$$\breve{x}_{Bb} = \frac{(1-\delta\ell)\left[(1-p_B)(h-\ell-c)-(p_G-p_B)(1-h)\lambda\right]-(1-\delta h)(1-p_B)(p_B-\ell)}{\delta(1-\ell)\left[(1-p_B)(h-\ell-c)-(p_G-p_B)(1-h)\lambda\right]-\delta(1-h)(1-p_B)(p_B-\ell)}$$

$$\breve{x}_{Gb} = \breve{x}_{Bb} - \frac{(p_G-p_B)(1-\delta h-\delta(1-h)\breve{x}_{Bb})}{\delta\left[(1-p_B)(h-\ell-c)-(p_G-p_B)(1-h)\lambda\right]}.$$

Note that $\breve{x}_{Bb}$ can be rewritten as

$$\breve{x}_{Bb} = \frac{(1-\delta\ell)(\overline{c}(\lambda,\delta)-c)}{(1-\delta\ell)(\overline{c}(\lambda,\delta)-c)+(1-\delta)(c-\widehat{c}(\lambda))}$$

$$\text{where} \quad \overline{c}(\lambda,\delta) = h-\ell-\frac{(1-\delta h)(p_B-\ell)}{(1-\delta\ell)}-\lambda(1-h)\frac{p_G-p_B}{1-p_B} \;>\; \widehat{c}(\lambda).$$

Hence, $\breve{x}_{Bb} \in (0,1)$ if and only if $\widehat{c}(\lambda) < c < \overline{c}(\lambda,\delta)$. In this case, the initial formula of $\breve{x}_{Bb}$ implies $(1-p_B)(h-\ell-c)-(p_G-p_B)(1-h)\lambda > 0$ and thus, $\breve{x}_{Gb} < \breve{x}_{Bb}$ and both $\breve{x}_{Bb}$ and $\breve{x}_{Gb}$ converge to 1 from below as $\delta \to 1$. This implies that $\breve{\mathbf{x}} = (1, 1, \breve{x}_{Bb}, \breve{x}_{Gb})$ supports a FSGV that dominates any FSGV with $x_{Gb} = 0$ required by possibility $(ii)$ and $(iii)$.

For $c \in (\widehat{c}(\lambda), \overline{c}(\lambda))$ and large enough $\delta$, therefore, $\overline{v}_F$ must be supported by a configuration $\mathbf{x}$ with $x_{Gb} > 0$ and moreover, ($\mathrm{ICh}_{BB}$) binds at $\mathbf{x}$ by [C2] because $\hat{\mathbf{x}}$ fails ($\mathrm{ICh}_{BB}$) in the current case. Since we may assume that (ICB) binds at a configuration that supports $\overline{v}_F$ by Lemma 1, we deduce that $\breve{\mathbf{x}} = (1, 1, \breve{x}_{Bb}, \breve{x}_{Gb})$ supports $\overline{v}_F$ for large enough $\delta$, which is calculated (by Mathematica) as

$$\overline{v}_F(\breve{\mathbf{x}}) = -\frac{c(1-\ell)}{h-\ell} - \frac{\left[\begin{array}{c}\ell(1-\lambda)\lambda^2+h^2(1-2\lambda)(\lambda^2-\lambda+(1-\ell)(1-2\lambda))\\ -h(1-4\lambda+5\lambda^2-\lambda^3-\ell(1-3\lambda+\lambda^2+2\lambda^3))\end{array}\right]}{(h-\ell)(1-\lambda)[h(1-2\lambda)^2(1-h)+\lambda(1-\lambda)]}h.$$

This establishes Proposition 2 for $\lambda > \widetilde{\lambda}$ and $\widehat{c}(\lambda) < c < \overline{c}(\lambda)$.

It remains to consider $c > \overline{c}(\lambda)$ for $\lambda > \widetilde{\lambda}$. If $\overline{v}_F$ is supported by $\mathbf{x}$ that conforms to possibility $(i)$ but $x_{Gb} > 0$ (hence, neither $(ii)$ and $(iii)$), then $\mathbf{x}$ must bind ($\mathrm{ICh}_{BB}$) by [C2] because $\hat{\mathbf{x}}$ fails ($\mathrm{ICh}_{BB}$), and $\mathbf{x}$ may also bind (ICB) by Lemma 1 but no such $\mathbf{x}$ exists as shown above.

Hence, it suffices to consider only $(ii)$ and $(iii)$. We may assume that (ICB) binds by Lemma 1. If ($\mathrm{ICh}_{BB}$) is slack, $x_{Bb} = 1$ must hold because otherwise raising $x_{Bb}$ would increase $v_F(\mathbf{x})$ maintaining (ICB) and (D), a contradiction; but $x_{Bb} = 1$ is inconsistent with $(ii)$ because binding (ICB) would imply $v_F(\mathbf{x}) - \underline{v} \to 0$ as $\delta \to 1$, contradicting ($\mathrm{ICh}_{BB}$), nor is it with $(iii)$ because it would imply $\Delta_b = 1 > \Delta_g$.

Therefore, $\overline{v}_F$ should be supported by a configuration that binds both (ICB) and ($\mathrm{ICh}_{BB}$), but we show this is impossible for large enough $\delta$ below, thus completing the proof of Proposition 2.

*Possibility (ii):* Suppose $x_{Gb} = 0$ and $x_{Bg} = 1$. Solving the simultaneous equation system consisting of the binding constraints (IC$h_{BB}$) and (ICB), and evaluating the solution value of $x_{Gg}$ at the limit $\delta = 1$ gives

$$\frac{(h-\ell-c)(1-\ell)-(p_G-p_B)(1-\ell)[h+\lambda-2h\lambda]-(p_B-\ell)[1-h\lambda-p_B(h+\lambda-2h\lambda)]}{(h-\ell-c)p_B(1-\ell)-(p_G-p_B)(1-\ell)h(1-\lambda)-(p_B-\ell)[h-h\lambda+p_B(1-2h-\lambda+2h\lambda)]}$$

which obtains a value of 1 at $c = \bar{c}(\lambda)$. Moreover, its derivative w.r.t. $c$ is

$$\frac{(1-\ell)[p_B(\lambda+h(1-2\lambda))-h(1-\lambda)][p_B^2-2\ell p_B+p_G(\ell-1)+\ell]}{\left((h-\ell-c)p_B(1-\ell)-(p_G-p_B)(1-\ell)h(1-\lambda)-(p_B-\ell)[h(1-\lambda)+p_B(1-2h-\lambda+2h\lambda)]\right)^2}$$

which is positive because i) $p_B(\lambda+h(1-2\lambda))-h(1-\lambda)<0$ given $p_B<h$ if $\lambda>\widetilde{\lambda}$ and ii) $p_B^2-2\ell p_B+p_G(\ell-1)+\ell<p_B^2-2\ell p_B+h(\ell-1)+\ell<0$ for $\ell<p_B<h$. Thus, we have shown that the unique solution value of $x_{Gg}$ to binding (IC$h_{BB}$) and (ICB) exceeds 1 for $c>\bar{c}(\lambda)$ and large enough $\delta$, hence no legitimate solution exists in the current case.

*Possibility (iii):* Suppose $x_{Gb} = 0$, $\Delta_b = \Delta_g$ and $x_{Bg} < 1$. Solving the simultaneous equation system consisting of the binding constraints (IC$h_{BB}$) and (ICB), and evaluating the solution value of $x_{Bg}$ at the limit $\delta = 1$ gives

$$x_{Bg}|_{\delta=1} = \frac{c+p_B+(p_G-p_B)(\lambda-\ell+2h(1-\lambda))-h}{c\ell+\ell(\ell-2h-(p_G-p_B)(1-\lambda-h+2h\lambda))+hp_G}. \tag{30}$$

The derivative of this w.r.t. $c$ is

$$\frac{\partial x_{Bg}|_{\delta=1}}{\partial c} = \frac{(h-\ell)(p_G(1-\ell)+p_B\ell-\ell)}{[c\ell+\ell(\ell-2h-(p_G-p_B)(1-\lambda-h+2h\lambda))+hp_G]^2}$$

which is positive because $p_G(1-\ell)+p_B\ell-\ell$ exceeds its value at $p_G = h$ and $p_B = \ell$, namely $(h-\ell)(1-\ell)$, given $\lambda>\widetilde{\lambda}$.

Moreover, we calculate that (30) has a value of 1 at

$$c = \frac{\ell(\ell-2h)-(p_G-p_B)(2h+(1-\ell)(1-2h)\lambda-\ell h)+h(1+p_G)-p_B}{1-\ell}.$$

Since subtracting this from $\bar{c}(\lambda)$ gives

$$\frac{(h(1-p_B)-\lambda(p_B(1-2h)+h))(p_G-p_B)}{1-p_B} = \frac{(1-2\lambda)(1-h)h(p_G-p_B)}{(1-h-\lambda(1-2h))(1-p_B)} > 0,$$

(30) obtains a value of 1 at some $c<\bar{c}(\lambda)$ and thus exceeds 1 for $c\geq\bar{c}(\lambda)$. Hence, no legitimate solution exists for $c>\bar{c}(\lambda)$ and large enough $\delta$.

## D Proof of Proposition 3

Consider a non-trivial self-generated value, SGV for short, for which the seller may mix $h$ and $\ell$ and/or report less than fully truthfully. For any message $m$ used in the period strategy supporting the SGV, associated are continuation values denoted by $v_{mq}$ for each $q \in \{g, b\}$. The "(continuation) spread" of message $m$ refers to $v_{mg} - v_{mb}$. Each $v_{mq}$ is between the SGV itself and $\underline{v}$, which we describe as being "feasible."

By an "agent e-s" we refer to a seller who exerted $e \in \{h, \ell\}$ and observed $s \in \{\mathbb{g}, \mathbb{b}\}$. Let $\pi_s^e = \pi_s$ if $e = h$ and $\pi_s^e = \pi_s'$ if $e = \ell$. For each message $m$ used for a SGV, the seller's payoff from sending $m$ is linear in $\pi_s^e$ with a slope equal to the spread$\times\delta$:

$$U(m, \pi_s^e) \ := \ (1-\delta)p_m + \delta[\pi_s^e(v_{mg} - v_{mb}) + v_{mb}].$$

Imagine the upper envelope of all the graphs of $U(m, \pi)$ for all $m$ on $\pi \in [\pi'_{\mathbb{b}}, \pi_{\mathbb{g}}]$. The optimal message(s) for each agent e-s are those whose graphs constitute the upper envelope at $\pi = \pi_s^e$. We may disregard any message whose graph is disjoint from the upper envelope. From the above, we have the following observations [i]–[v] on the seller behavior supporting a SGV, that hold for all $\lambda \in (0, 1/2)$.

[i] If agents e-s and e'-s' find it optimal to send $m$ and $m'$, respectively, then the spread of $m$ is weakly larger than that of $m'$ if $\pi_s^e \geq \pi_{s'}^{e'}$ from the discussion above, or equivalently, because

$$\begin{aligned}
(1-\delta)p_m + \delta[\pi_s^e v_{mg} + (1-\pi_s^e)v_{mb}] &\ \geq\ (1-\delta)p_{m'} + \delta[\pi_s^e v_{m'g} + (1-\pi_s^e)v_{m'b}] \\
(1-\delta)p_m + \delta[\pi_{s'}^{e'} v_{mg} + (1-\pi_{s'}^{e'})v_{mb}] &\ \leq\ (1-\delta)p_{m'} + \delta[\pi_{s'}^{e'} v_{m'g} + (1-\pi_{s'}^{e'})v_{m'b}] \\
\Longrightarrow\ \pi_s^e[(v_{mg} - v_{mb}) - (v_{m'g} - v_{m'b})] &\ \geq\ \pi_{s'}^{e'}[(v_{mg} - v_{mb}) - (v_{m'g} - v_{m'b})].
\end{aligned}$$

[ii] Each used message is optimal for an "adjacent" set of agents, i.e., all agents with $\pi_s^e$ in a certain interval. Multiple messages optimal for multiple agents must have the same graph, hence same spread, $v_{mg} - v_{mb}$, and same intercept, $(1-\delta)p_m + \delta v_{mb}$.

[iii] Any two messages $m$ and $m'$ optimal for all agents in a given set of agents can be replaced, without affecting optimality conditions, by a new message obtained by the convex combination of $m$ and $m'$ with weights equal to their respective probabilities relative to total probability. Hence, one may assume at most one message that is optimal for and only for all agents in any given "adjacent" subset of agents.[20] Moreover, if such a message exists for an adjacent set of agents, then it is the unique message commonly optimal for any non-singleton subset of those agents.

[20] Note that this doesn't prevent that several messages be optimal for any given agent.

[iv] If $h$ is exerted with positive probability, the spread of any message optimal for the agent $h$-$\mathbb{g}$ is positive: otherwise, the spread of every used message would be non-positive and thus, the upper envelope of the graphs of all messages is non-positively sloped. This would mean that the optimal expected payoff is no lower for an $\ell$-seller than for an $h$-seller so that exerting $h$ is suboptimal considering the cost $c > 0$, contrary to $h$ being exerted.

[v] Consider two used messages $m$ and $m'$ with associated prices $p_m$ and $p_{m'}$, respectively, such that $p_m < p_{m'}$ and the spread of $m'$ is larger than that of $m$ which is positive. Consider a third message $m''$ with $p_m < p_{m''} < p_{m'}$ followed by continuation values $v_{m''q} = v_{mq} - (1-\delta)(p_{m''} - p_m)/\delta$ for $q \in \{g, b\}$. Then, the expected payoff from sending $m$ and $m''$ are identical for all agents and $v_{m''q}$ is feasible. It is feasible because the agent who uses $m$ must weakly prefer sending $m''$ to $m'$, which implies, together with both the price and the spread being larger for $m'$ than $m''$, that $v_{m''q} \geq v_{m'b}$.

With these observations at hand, we now show that there is no SGV larger than $\max\{v^*, \overline{v}_F\}$ in Step 1 and Step 2 below. Then, we show that all values in $[\underline{v}, \max\{v^*, \overline{v}_F\}]$ can be achieved as equilibrium values in Step 3.

<u>*STEP 1*: Not fully truthful announcements do not increase the seller's value</u>

Consider a SGV $v_h (> \underline{v})$ for which the seller exerts $h$ for sure but does not report the signal fully truthfully. We show that such a SGV is no higher than $\max\{v^*, \overline{v}_F\}$ if $c < h - \ell$ and $\delta$ is large enough.

By observation [iii], we may consider only up to three used messages with at most one of them being sent by both agent h-$\mathbb{g}$ and agent h-$\mathbb{b}$. Hence, there are three possibilities to consider as below.

<u>Possibility 1:</u> Suppose that an $h$-seller always announces a message $G$ after $s = \mathbb{g}$, but after $s = \mathbb{b}$ she announces $G$ and $B$ with probability $\theta \in (0, 1)$ and $1 - \theta$, respectively. We call the associated SGV a *semi-faithfully* SGV. Then, the price remains at $p_B$ after $m = B$ but changes to

$$p_G^\theta \;:=\; p_G - \frac{\kappa(\theta)}{h(1-\lambda) + (1-h)\lambda} \;>\; h \;>\; p_B \;>\; \ell$$

after $m = G$ where

$$\kappa(\theta) \;:=\; \frac{h(1-h)(1-2\lambda)\theta}{h(1-\lambda) + (1-h)\lambda + (h\lambda + (1-h)(1-\lambda))\theta} < h(1-h)$$

and the value (computed from truthful reporting out of indifference) is

$$v_h(\mathbf{x}) = \frac{(1-\delta)(h - \ell - c - \kappa(\theta))}{1 - \delta\left(h\rho_g + (1-h)\rho_b\right)} + \underline{v}$$

We assume $h - \ell - c - \kappa(\theta) > 0$ because $v_h(\mathbf{x}) > \underline{v}$.

The maximum semi-faithfully SGV, which we denote by $\overline{v}_h$ if exists, is the solution value to the program

$$\overline{v}_h \;=\; \max_{\mathbf{x}\in[0,1]^4} \; v_h(\mathbf{x}) \tag{31}$$

subject to (ICh$_{GB}$), (ICh$_{BB}$), (ICB) and (D) with $p_G$ replaced by $p_G^\theta$ and $v_F(\mathbf{x})$ by $v_h(\mathbf{x})$, and the additional restriction that (ICB) binds at $\mathbf{x}$.

In the proof of Lemma 1, to prove that $\overline{v}_F$ is supported by a configuration that binds (ICB), the actual value of $p_G$ was not used but only the fact that $p_G \in (h, 1)$. Therefore, Lemma 1 extends to establish that the maximum semi-faithfully SGV, $\overline{v}_h$, is also the solution value to the program (31) *without* requiring that (ICB) bind at $\mathbf{x}$, which is how we treat (31) from now on. Then, various results on $\overline{v}_F$ extend to $\overline{v}_h$ as described below.

We start with the observation that $\overline{v}_h$ is dominated by $\overline{v}_F$ for large enough $\delta$ if the latter also exists:

**Lemma 3** *If $c < \bar{c}(\lambda)$ then $\overline{v}_h \le \overline{v}_F$ for large enough $\delta$ if $\overline{v}_h$ exists.*

*Proof.* Consider a maximum semi-faithfully SGV $\overline{v}_h$ supported by a configuration $\mathbf{x}$ such that $x_{Gb} \to 1$ as $\delta \to 1$ (because otherwise $\overline{v}_h < \overline{v}_F$ for large enough $\delta$ by Proposition 2). By observations [i]–[v] above, the payoff from sending $m = B$ is $U(B,\pi) = (1-\delta)p_B + \delta[\pi(v_{Bg} - v_{Bb}) + v_{Bb}]$ and that from sending $m = G$ is $U(G,\pi) = (1-\delta)p_G^\theta + \delta[\pi(v_{Gg} - v_{Gb}) + v_{Gb}]$ such that $U(B,\pi_\mathbb{b}) = U(G,\pi_\mathbb{b})$ and $v_{Gg} - v_{Gb} > \max\{0, v_{Bg} - v_{Bb}\}$. Since $x_{Gb} \to 1 \;\Leftrightarrow\; v_{Gb} \to \overline{v}_h$ as $\delta \to 1$, one can find $v'_{Gb} \in (\underline{v}, v_{Gb})$ such that

$$(1-\delta)(\pi_\mathbb{g} - p_G^\theta) = \delta(1-\pi_\mathbb{g})(v_{Gb} - v'_{Gb})$$

so that $(1-\delta)\pi_\mathbb{g} + \delta[\pi(v_{Gg} - v'_{Gb}) + v'_{Gb}]$ is equal to $U(G,\pi)$ at $\pi = \pi_\mathbb{g}$ but lower than $U(G,\pi)$ at $\pi < \pi_\mathbb{g}$. Hence, when $p_G^\theta$ is replaced by $p_G = \pi_\mathbb{g}$ and $v_{Gb}$ by $v'_{Gb}$, the seller would find the faithful strategy optimal and thus, $\overline{v}_h$ can be generated as a FSGV. ∎

By the same reasoning as in [C1] of the proof of Proposition 2, (ICB) must hold at $(1,1,1,0)$ for any $\mathbf{x}$ to bind (ICB), which is the case for large $\delta$ only if

$$c \;<\; c_{ICB}^\theta := h - \ell - \kappa(\theta) - \lambda(1-h)\frac{p_G^\theta - p_B}{1-p_B} \;<\; c_{ICB}$$

where the latter inequality follows from

$$-\kappa(\theta) + \frac{\lambda(1-h)\kappa(\theta)}{(1-p_B)(h(1-\lambda)+(1-h)\lambda)} < 0 \tag{32}$$

because $\frac{\lambda(1-h)}{(1-p_B)(h(1-\lambda)+(1-h)\lambda)} < \frac{\lambda(1-h)}{(1-p_B)(h(1-\lambda)+(1-h)\lambda)}\Big|_{p_B=h} < 1$. In addition, for $\lambda \le \widetilde{\lambda}$, analogously to the first paragraph of Case C.1 in the proof of Proposition 2, (ICh$_{GB}$) may hold for some $\mathbf{x}$ only if

$$c \ < \ c^* + (1-2\lambda)\frac{(h-\ell)(1-h)}{1-\ell}(p_G^\theta - p_B) - \kappa(\theta)\frac{h-\ell}{1-\ell} \ < \ c_{ICh}.$$

Hence, for $\lambda \leq \widetilde{\lambda}$, if a semi-faithfully SGV exists then $c < \overline{c}(\lambda)$ and thus, $\overline{v}_h \leq \overline{v}_F$ by Lemma 3.

Next, consider the case that $\lambda > \widetilde{\lambda}$. Assume $c < c^\theta_{ICB}$ so that (ICB) is satisfied at some $\mathbf{x}$, thus at $\mathbf{x} = (1,1,1,0)$. Again, the arguments for the Case C.2 in the proof of Proposition 2 extend to the current case with $p_B$ replaced by $p^\theta_B$ and $v_F(\mathbf{x})$ by $v_h(\mathbf{x})$. In particular, (ICB) holding at $\mathbf{x} = (1,1,1,0)$ implies that $v_h(\mathbf{x})$ is maximized subject to (ICB) and (D) at $\hat{\mathbf{x}}^\theta = (1,1,1,\hat{x}^\theta_{Gb})$ that binds (ICB). Since (ICh$_{BB}$) is satisfied at $\hat{\mathbf{x}}^\theta$ if and only if

$$c \ \leq \ \widehat{c}^\theta(\lambda) = h - p_B - \kappa(\theta) - \lambda(1-h)\frac{p^\theta_G - p_B}{1-p_B} < \widehat{c}(\lambda)$$

where the latter inequality is due to (32), it follows that $\bar{v}_h = v_h(\hat{\mathbf{x}}^\theta)$ if $c \leq \widehat{c}^\theta(\lambda)$, in which case $\overline{v}_h \leq \overline{v}_F$ by Lemma 3 because $\widehat{c}(\lambda) < \overline{c}(\lambda)$ for $\lambda > \widetilde{\lambda}$.

In addition, for $c > \widehat{c}^\theta(\lambda)$, the arguments in Case C.2 establish that (28) and (29) must hold at the solution $\mathbf{x}$ to (31), leaving three cases to consider: $(i)$ $x_{Bg} = x_{Gg} = 1$, $(ii)$ $x_{Gb} = 0$ and $x_{Bg} = 1$, $(iii)$ $x_{Gb} = 0,\ \Delta_b = \Delta_g$ and $x_{Bg} < 1$. The analyses for these cases also extend straightforwardly with suitable modifications as summarized below. Used in this process is the claim [C2] which is straightforwardly verified to hold for $\overline{v}_h$ as well.

For the case $(i)$ $x_{Bg} = x_{Gg} = 1$, the solution values $\breve{x}_{Bb}$ and $\breve{x}_{Gb}$ that bind both (ICB) and (ICh$_{BB}$) are of the same formulae as before with $c$ replaced by $c + \kappa(\theta)$ and $p_G$ by $p^\theta_G$, thus a legitimate solution exists only if

$$c \ < \ \overline{c}^\theta(\lambda) = h - \ell - \kappa(\theta) - \frac{(1-\delta h)(p_B - \ell)}{(1-\delta\ell)} - \lambda(1-h)\frac{p^\theta_G - p_B}{1-p_B} < \overline{c}(\lambda)$$

where the inequality follows from (32). In this case, (ICh$_{BB}$) must bind at a configuration supporting $\overline{v}_h$ by [C2], where (ICB) also binds by Lemma 1. Thus, $(1,1,\breve{x}_{Bb},\breve{x}_{Gb})$ supports $\overline{v}_h$ if $c \in (\widehat{c}^\theta(\lambda), \overline{c}^\theta(\lambda))$, whence $\overline{v}_h \leq \overline{v}_F$ by Lemma 3.

For the remaining case that $\lambda > \widetilde{\lambda}$ and $c \geq \overline{c}^\theta(\lambda)$, we may focus on possibilities $(ii)$ and $(iii)$ and (ICh$_{BB}$) should bind at the configuration that supports $\overline{v}_h$ for the same reasoning as in Case C.2 of proof of Proposition 2, where we may assume (ICB) also binds by Lemma 1. But, such a configuration does not exist for large enough $\delta$ as asserted below.

For the possibility $(ii)$ $x_{Gb} = 0$ and $x_{Bg} = 1$, the solution value $x_{Gg}|_{\delta=1}$ obtains a value of 1 at $c = \overline{c}^\theta(\lambda)|_{\delta=1} < \overline{c}(\lambda)$ and increases in $c$ because its derivative w.r.t. $c$ is of the same formula as in C.2 with $c$ replaced by $c + \kappa(\theta)$ and $p_G$ by $p^\theta_G$. Hence, no legitimate solution exists. For $(iii)$ $x_{Gb} = 0,\ \Delta_b = \Delta_g$ and $x_{Bg} < 1$, again the suitably modified solution value of $x_{Bg}$ increases in $c$ and exceeds 1 at $c = \overline{c}^{\theta}(\lambda)$ by the same reasoning, precluding any legitimate solution.

Possibility 2: Suppose that a SGV $v_h$ is supported by an $h$-seller who always announces a message $B$ after $s = \mathbb{b}$, but announces $G$ and $B$ with probability $\theta \in (0,1)$ and $1-\theta$, respectively, after $s = \mathbb{g}$. Then, the price remains at $p_G$ after $m = G$ while it is $p_B^{\theta} \in (p_B, h)$ after $m = B$.

As exerting $h$ and reporting $B$ is optimal for the seller, her value is

$$\begin{aligned} v_h &= (1-\delta)(p_B^{\theta} - c) + \delta(hx_{Bg} + (1-h)x_{Bb})(v_h - \underline{v}) + \delta\underline{v} \\ \Longrightarrow \quad v_h &= \frac{(1-\delta)(p_B^{\theta} - \ell - c)}{1 - \delta(hx_{Bg} + (1-h)x_{Bb})} + \underline{v} \end{aligned}$$

so that, in particular, we need $p_B^{\theta} > \ell$ for $v_h > \underline{v}$. As an $\ell$-seller could always report $B$, optimality of exerting $h$ requires

$$\delta(h-\ell)(x_{Bg} - x_{Bb})(v_h - \underline{v}) \ \geq \ (1-\delta)c$$

subject to which $v_h$ is maximized at $x_{Bg} = 1$ and $x_{Bb}$ that binds the inequality.

Notice from (ICh), however, that this is the condition for $(e_1, \rho_g, \rho_b)$ to constitute a self-generated value without communication where $e_1$ satisfies $p_B^{\theta} = e_1(h-\ell) + \ell$ and $(\rho_g, \rho_b) = (x_{Bg}, x_{Bb})$. As shown in Section 2, therefore, $v_h \leq v^*$ if $c < c^*$ and $v_h = \underline{v}$ if $c \in [c^*, h-\ell)$.

Possibility 3: The remaining possibility is that an $h$-seller sends $G$ and a third message $m$ after $s = \mathbb{g}$, and $B$ and $m$ after $s = \mathbb{b}$ for a SGV. Then, $p_B < p_m < p_G$ and we must have

$$\begin{aligned} (1-\delta)p_B + \delta[\pi_{\mathbb{b}} v_{Bg} + (1-\pi_{\mathbb{b}})v_{Bb}] &= (1-\delta)p_m + \delta[\pi_{\mathbb{b}} v_{mg} + (1-\pi_{\mathbb{b}})v_{mb}] \\ &\geq (1-\delta)p_G + \delta[\pi_{\mathbb{b}} v_{Gg} + (1-\pi_{\mathbb{b}})v_{Gb}] \\ \text{and} \quad (1-\delta)p_G + \delta[\pi_{\mathbb{g}} v_{Gg} + (1-\pi_{\mathbb{g}})v_{Gb}] &= (1-\delta)p_m + \delta[\pi_{\mathbb{g}} v_{mg} + (1-\pi_{\mathbb{g}})v_{mb}] \\ &\leq (1-\delta)p_B + \delta[\pi_{\mathbb{g}} v_{Bg} + (1-\pi_{\mathbb{g}})v_{Bb}]. \end{aligned}$$

Therefore, if both $G$ and $B$ are sent with positive probability, the same value is generated by $h$-seller who sends $G$ and $B$ with certainty after $s = \mathbb{g}$ and $s = \mathbb{b}$, respectively, i.e., through a faithful strategy, with the same continuation values. Note that the seller cannot benefit by exerting $\ell$ instead of $h$ with message $m$ removed, because the expected payoffs from sending $G$ and $B$ remain the same for $\ell$-seller. If both $G$ and $B$ are unused, on the other hand, it amounts to babbling and SGV's of this kind have been covered in Section 2. The case that only $G$ or only $B$ is unused amounts to Possibility 1 and Possibility 2 above, respectively.

STEP 2: Mixing $h$ and $\ell$ does not increase the seller's value, either.

We will show that for any SGV for which the seller mixes $h$ and $\ell$, there is a weakly higher SGV for which effort is not mixed.

**Lemma 4** *Consider a SGV $v' > \underline{v}$ for which $h$ and $\ell$ are mixed.*
*(a) If a message $G'$ is used by agent $h$-$\mathbb{g}$ with $p_{G'} \leq \pi_{\mathbb{g}}$ and $B' \neq G'$ is used by agent $h$-$\mathbb{b}$ with $p_{B'} \leq \pi_{\mathbb{b}}$, then there is a weakly higher SGV for which effort is not mixed.*
*(b) If a message $G'$ is used by agent $\ell$-$\mathbb{g}$ and $B' \neq G'$ is used by agent $\ell$-$\mathbb{b}$, then either $p_{G'} > \pi'_{\mathbb{g}}$ or $p_{B'} > \pi'_{\mathbb{b}}$.*

*Proof.* Part (a). Consider a SGV $v' > \underline{v}$ as above. We build a weakly higher SGV supported by a period strategy in which $h$ is exerted for sure, and alternative messages $G''$ and $B''$ with prices $p_{G''} \leq \pi_{\mathbb{g}}$ and $p_{B''} = \pi_{\mathbb{b}}$, together with suitably modified continuation values. Recall that payoff from sending message $m$ depends only on $(1-\delta)p_m + \delta v_{mb}$ and the spread $v_{mg} - v_{mb}$ (observation [i]).

If $p_{G'} = \pi_{\mathbb{g}}$ and $p_{B'} = \pi_{\mathbb{b}}$, then $v'$ can be supported by the seller exerting $h$ for sure and sending $G'$ ($B'$) after good (bad) signal, i.e., without mixing $h$ and $\ell$. Hence, suppose $p_{G'} < \pi_{\mathbb{g}}$ or $p_{B'} < \pi_{\mathbb{b}}$, and consider hypothetical messages denoted by $G'_d$ and $B'_d$ with associated prices higher than $p_{G'}$ and $p_{B'}$, respectively, by the same amount $d > 0$, keeping the continuation values unchanged. Increase $d$ until either $p_{B'_d}$ hits $\pi_{\mathbb{b}}$ or $p_{G'_d}$ hits $\pi_{\mathbb{g}}$.

Case 1. $\pi_{\mathbb{g}} - p_{G'} \geq \pi_{\mathbb{b}} - p_{B'}$: In this case, $p_{B'_d}$ hits $\pi_{\mathbb{b}}$, say at $d = d1$, before $p_{G'_d}$ hits $\pi_{\mathbb{g}}$. Keep increasing $p_{G'}$ while decreasing $\frac{\delta}{1-\delta} v_{G'_d g}$ and $\frac{\delta}{1-\delta} v_{G'_d b}$ by the same amount to keep $G'_d$ to be "equivalent" with $G'_{d1}$ for every agent. If $p_{G'_d}$ reaches $\pi_{\mathbb{g}}$, say at $d = d2$, before $v_{G'_d b}$ hits $\underline{v}$ (recall that $v_{G'b} < v_{G'g}$ from observation [iv]), we have found two messages $G'' = G'_{d2}$ and $B'' = B'_{d1}$ with $p_{G''} = \pi_{\mathbb{g}}$ and $p_{B''} = \pi_{\mathbb{b}}$, such that exerting $h$ and announcing $G''$ if $s = \mathbb{g}$ and $B''$ if $s = \mathbb{b}$ is at least as good as any other strategy. Thus a SGV $v'' > v'$ is supported by a seller exerting $h$ for sure (because prices are higher).

If $v_{G'_d b}$ hits $\underline{v}$ before $p_{G'_d}$ reaches $\pi_{\mathbb{g}}$, then at that point, given the two messages $B'_{d1}$ and $G'_d$, the seller finds it optimal to exert $h$ and report $B'_{d1}$ after $s = \mathbb{b}$ and $G'_d$ after $s = \mathbb{g}$, which generates a value $v > v'$ given by

$$v = -(1-\delta)c + (h(1-\lambda) + (1-h)\lambda)\big[(1-\delta)p_{G'_d} + \delta\pi_{\mathbb{g}}(v_{G'_d g} - v_{G'_d b}) + \delta v_{G'_d b}\big]$$
$$+(h\lambda + (1-h)(1-\lambda))\big[(1-\delta)\pi_{\mathbb{b}} + \delta\pi_{\mathbb{b}}(v_{B'_{d1} g} - v_{B'_{d1} b}) + \delta v_{B'_{d1} b}\big]$$

and all continuation values are "feasible", i.e., between $\underline{v}$ and $v'$. Note also that $p_{G'_d} > p_{G'} > \pi_{\mathbb{b}}$ because $G'$ is sent by no agent $e$-$s$ with $\pi^e_s < \pi_{\mathbb{b}}$ to self-generate $v'$ initially (observation [iii]).

Now, increase $p_{G'_d}$ until either it reaches $\pi_{\mathbb{g}}$ or agent $h$-$\mathbb{b}$ becomes indifferent between $B'_{d1}$ and $G'_d$. At that point the value is $v'' > v'$. In the case that $p_{G'_d}$ reaches $\pi_{\mathbb{g}}$ first, $v''$

is a SGV supported by a seller exerting $h$ for sure and sending $B'' = B'_{d1}$ after $s = \mathbb{b}$ and $G'' = G'_d$ after $s = \mathbb{g}$ (exerting $h$ is optimal because an $\ell$-seller benefits less from increased $p_{G'_d}$ given that it announces $B''$ if $s = \mathbb{b}$ and thus announces $G''$ less often than an $h$ -seller).

If $p_{G'_d} < \pi_{\mathbb{g}}$ reaches a point at which agent $h$-$\mathbb{b}$ becomes indifferent between $B'_{d1}$ and $G'_d$, then there is a higher SGV $v''$ obtained with agent $h$-$\mathbb{b}$ mixing between $B'' = B'_{d1}$ and $G'' = G'_d$ appropriately so that $p_{G'_d}$ is the price for the message $G'_d$ obtained by Bayes rule. This higher SGV is thus supported by a seller exerting $h$ for sure and mixing as such after $s = \mathbb{b}$ while sending $G'_d$ for sure after $s = \mathbb{g}$ (exerting $h$ is optimal for the seller for the same reason as above.)

Case 2. $\pi_{\mathbb{g}} - p_{G'} < \pi_{\mathbb{b}} - p_{B'}$: Next, consider the case that $p_{G'_d}$ hits $\pi_{\mathbb{g}}$ first at $d = d1$. If the spread of $B'$ is positive, then use observation [v] to replace message $B'_{d1}$ by an equivalent message $B''$ with price $p_{B''} = \pi_{\mathbb{b}}$ (use $m = G'_{d1},\ m' = B'_{d1}$ and $p_{m''} = \pi_{\mathbb{b}}$ in [v]). Then the SGV $v'$ can be supported with effort $h$ only and messages $G'' = G'_{d1}$ and $B''$.

If the spread of $B'$ is negative, consider a modified message $B'_d$ by increasing $p_{B'_d}$ toward $\pi_{\mathbb{b}}$ while decreasing $\delta v_{B'_d g}/(1-\delta)$ and $\delta v_{B'_d b}/(1-\delta)$ by the same amount to keep $B'_d$ to be "equivalent" with $B'_{d1}$ for every agent. If $p_{B'_d}$ reaches $\pi_{\mathbb{b}}$ before $v_{B'_d g}$ hits $\underline{v}$, then a higher SGV is supported by a seller exerting $h$ for sure and reporting $B'_d$ when $s = \mathbb{b}$ and and $G'_{d1}$ when $s = \mathbb{g}$, by the same reasoning as above.

In the alternative case that $v_{B'_d g}$ hits $\underline{v}$ before $p_{B'_d}$ reaches $\pi_{\mathbb{b}}$, denote the message at that point by $B'_{d2}$ for later reference. Continue to increase $p_{B'_d}$ to $\pi_{\mathbb{b}}$ while decreasing $v_{B'_d b}$ to keep the payoff of agent $h$-$\mathbb{b}$ from sending $B'_d$, i.e. $(1-\delta)p_{B'_d} + \delta\pi_{\mathbb{b}}(\underline{v} - v_{B'_d b}) + \delta v_{B'_d b}$, constant. Note that $p_{B'_d}$ reaches $\pi_{\mathbb{b}}$ before $v_{B'_d b}$ hits $\underline{v}$, because otherwise the payoff of agent $h$-$\mathbb{b}$ from sending $B'$ would be less than $(1-\delta)\pi_{\mathbb{b}} + \delta\underline{v}$ which is strictly less than that from sending $G'$.

At this stage, points i) and ii) below establish that exerting $h$ and announcing $B'_d$ after $s = \mathbb{b}$ and $G'_{d1}$ after $s = \mathbb{g}$ is optimal.

i) Suppose that exerting $h$ is optimal given messages $B'_d$ and $G'_{d1}$. Then, since the spread of $B'_d$ is negative, the payoff of agent $h$-$\mathbb{g}$ from sending $G'_{d1}$ must exceed that of agent $\ell$-$\mathbb{g}$ from sending $B'_d$, which further implies that it is optimal for agent $h$-$\mathbb{g}$ to send $G'_{d1}$. Thus, a higher SGV is supported by a seller exerting $h$ for sure and sending $B'_d$ after $s = \mathbb{b}$ while sending $G'_{d1}$ after $s = \mathbb{g}$. If $\lambda \geq \widetilde{\lambda}$, exerting $\ell$ gets worse after $B'_{d2}$, so exerting $h$ has to be optimal.

ii) To show that exerting $\ell$ cannot be uniquely optimal given messages $B'_d$ and $G'_{d1}$ when $\lambda < \widetilde{\lambda}$, suppose the contrary. Then, $B'_d$ must be optimal for agent $\ell$-$\mathbb{g}$ because otherwise, as the process from $B'_{d2}$ to $B'_d$ reduced the payoff of agent $\ell$-$\mathbb{b}$ (from sending $B'_d$) while keeping the payoff of agent $\ell$-$\mathbb{g}$ intact (from sending $G'_{d1}$), $\ell$ would be a worse option than $h$. Moreover, for the expected payoff from exerting $\ell$ and sending $B'_d$ regardless of the signal, $(1-\delta)p_{B'_d} + \delta\ell(\underline{v} - v_{B'_d b}) + \delta v_{B'_d b}$, to increase as $B'_{d2}$ changes to $B'_d$ (during which we have $(1-\delta)dp_m + \delta(1-\pi_{\mathbb{b}})dv_{mb} = 0$) we need $\delta(\pi_{\mathbb{b}} - \ell)dv_{mb} > 0$ which implies $\pi_{\mathbb{b}} < \ell$. Thus, the SGV $v\,(> \underline{v})$ would have to satisfy

$$v \leq (1-\delta)\ell + \delta\ell(\underline{v} - v) + \delta v \leq (1-\delta)\ell + \delta v \Leftrightarrow \ v \leq \ell = \underline{v},$$

which is a contradiction. Hence, exerting $h$ must be optimal.

Part $(b)$. If $p_{G'} \leq \pi'_{\mathbb{g}}$ and $p_{B'} \leq \pi'_{\mathbb{b}}$, the value $v'$ from optimally exerting $\ell$ and sending messages $G'$ and $B'$ after respective signals is at most

$$(1-\delta)\big[(\ell(1-\lambda) + (1-\ell)\lambda)\pi'_{\mathbb{g}} + (\ell\lambda + (1-\ell)(1-\lambda))\pi'_{\mathbb{b}}\big] + \delta v' \ \leq (1-\delta)\ell + \delta v',$$

leading to $v' \leq \ell = \underline{v}$, a contradiction. ■

We now consider an arbitrary SGV, $v' > \underline{v}$, for which $h$ and $\ell$ are mixed and show that there is a weakly higher SGV supported by a strategy where effort is not mixed, if $\delta < 1$ is large enough. If no message is shared between $h$-seller and $\ell$-seller, then $v'$ is clearly supported by the behavior of the $h$-seller alone, i.e., by a period strategy of exerting $h$ for sure. Thus, we suppose that some messages are sent by both $h$-seller and $\ell$-seller. By observation [ii], there are four cases to be analyzed as below.

*CASE 1: There is no message optimal for three or more agents for $v'$.*

If $\lambda \geq \widetilde{\lambda}$, agents $h$-$\mathbb{b}$ and $\ell$-$\mathbb{g}$ must share a message between them, say $B'$ with $p_{B'} < \pi_{\mathbb{b}}$, and agent $h$-$\mathbb{g}$ use another message $G'$ with $p_{G'} \leq \pi_{\mathbb{g}}$, so that Lemma 4 $(a)$ applies.

Hence, we consider $\lambda < \widetilde{\lambda}$ below. Then, there is a message $G'$ used by agent $h$-$\mathbb{g}$ with $p_{G'} \in (\pi'_{\mathbb{g}}, \pi_{\mathbb{g}}]$ and another message $B'$ used by $h$-$\mathbb{b}$ with $p_{B'} < \pi'_{\mathbb{g}}$. Let $B'$ be the message with lowest price used by $h$-$\mathbb{b}$. If $p_{B'} \leq \pi_{\mathbb{b}}$, apply Lemma 4 $(a)$. The other case, $p_{B'} > \pi_{\mathbb{b}}$, is impossible for then it would follow that $B'$ is used by the agent $\ell$-$\mathbb{g}$ as well and the agent $\ell$-$\mathbb{b}$ has a sole-use message, say $B''$, contradicting Lemma 4 $(b)$ as $p_{B'} < \pi'_{\mathbb{g}}$ and $p_{B''} = \pi_{\mathbb{b}}$.

*CASE 2: There is a message $H$ optimal for all agents but $\ell$-$\mathbb{b}$.*

If $p_H \leq h$, a higher SGV is supported by a seller exerting $h$ for sure and sending a modified message $H'$ from $H$ by increasing the price to $p_{H'} = h$ with the same continuation values. This is the case if $\lambda > \widetilde{\lambda}$ because then agent $h$-$\mathbb{b}$ sends $H$ for sure by observation [iii] and thus, $p_H$ cannot exceed $h$.

Suppose $p_H > h$ so that $\lambda \leq \widetilde{\lambda}$. (If $\lambda = \widetilde{\lambda}$ and there are multiple optimal messages for agents $h$-$\mathbb{b}$ and $\ell$-$\mathbb{g}$, then we may assume w.l.o.g. that both agents use all optimal messages with positive probability.) If there is $B' \neq H$ optimal for $h$-$\mathbb{b}$ such that $p_{B'} \leq \pi_{\mathbb{b}}$, then apply Lemma 4 $(a)$.

Otherwise, agents $h$-$\mathbb{b}$ and $\ell$-$\mathbb{g}$ send only $H$ by observation [iii] so that $p_H < \pi'_{\mathbb{g}}$ (because $p_H > h$) and there is a message solely used by $\ell$-$\mathbb{b}$, say $B''$ with $p_{B''} = \pi'_{\mathbb{b}}$. But this contradicts Lemma 4 $(b)$.

*CASE 3: There is a message $L$ optimal for all agents but $h$-$\mathbb{g}$.*

If $p_L \leq \ell$, a higher SGV is supported by a seller exerting $\ell$ for sure and sending a modified message $L'$ from $L$ by increasing the price to $p_{L'} = \ell$.

Suppose $p_L > \ell$. If $p_L \leq \pi_{\mathbb{b}}$, which has to be the case if $\lambda \geq \widetilde{\lambda}$, apply Lemma 4 ($a$). If $p_L > \pi_{\mathbb{b}}$, so that $\lambda < \widetilde{\lambda}$, then there is $G'$ used by $h$-$\mathbb{g}$ with $p_{G'} > \pi'_{\mathbb{g}}$ by [iii] above, and there is a message $m$ solely used by $\ell$-$\mathbb{b}$ because otherwise we would have $p_L \leq \max\{\ell, \pi_{\mathbb{b}}\}$. Since $p_m = \pi'_{\mathbb{b}}$, Lemma 4 ($b$) dictates that $p_L > \pi'_{\mathbb{g}}$ which is a contradiction because $L$ is not used by agent $h$-$\mathbb{g}$.

*CASE 4: There is a message $M$ used by all agents.*

If $p_M \leq h$, which has to be the case if $\lambda \geq \widetilde{\lambda}$, a higher SGV is supported by a seller exerting $h$ for sure and sending a modified message $M'$ from $M$ by increasing the price to $p_{M'} = h$.

If $p_M > h$, so that $\lambda < \widetilde{\lambda}$, then there must be a message $B''$ solely used by $\ell$-$\mathbb{b}$ with $p_{B''} = \pi'_{\mathbb{b}}$ and moreover, $p_M < \pi'_{\mathbb{g}}$. This contradicts Lemma 4 ($b$).

<u>*STEP 3*: every value in $[\underline{v}, \max\{v^*, \overline{v}_F\}]$ is an equilibrium value</u>

In STEPs 1-2 above, we have shown that $\max\{v^*, \overline{v}_F\}$ is the tight upper bound of all SGV for large enough $\delta < 1$. When $v^* = \max\{v^*, \overline{v}_F\}$, the claim has been shown already in Proposition 1.

Hence, consider the alternative case, i.e., $\overline{v}_F > v^*$, implying that $c < \overline{c}(\lambda)$. From the proof of Proposition 2, there exists $v_0 < \overline{v}_F$ such that a continuum $[v_0, \overline{v}_F]$ of FSGV's exists for large enough $\delta$.

Consider the following strategy: exert $\ell$ in the first $t$ periods, followed by a continuation value $v \in [v_0, \overline{v}_F]$, generating a value of $\ell(1-\delta^t) + \delta^t v = \ell(1-\delta^t) + \delta^t(v - \underline{v}) + \delta^t \underline{v} = \underline{v} + \delta^t(v - \underline{v})$. Given the price of $\ell$ in the first $t$ periods followed by such a continuation value $v$, it is optimal to exert $\ell$ in the first $t$ periods. The set of values that can be generated with $t$ lags as such is $[\underline{v} + \delta^t(v_0 - \underline{v}), \underline{v} + \delta^t(\overline{v}_F - \underline{v})]$ . Note that $\underline{v} + \delta^{t+1}(\overline{v}_F - \underline{v}) - \underline{v} - \delta^t(v_0 - \underline{v}) = \delta^t(\delta\overline{v}_F - v_0 + \underline{v} - \delta\underline{v}) > 0$ where the inequality holds if $\delta$ is large enough. Therefore, every value in $(\underline{v}, v_0)$ is a FSGV with $t$ lags for some $t$ if $\delta$ is large enough. Consequently, every value in $(\underline{v}, \overline{v}_F]$ is either a FSGV or a FSGV with $t$ lags for some $t$ if $\delta$ is large enough, all of which constitute equilibrium values (with public randomization).

This proves Step 3, thus completing the proof of Proposition 3.

# E Proof of Proposition 4

From the findings in the paragraph preceding Proposition 4, we deduce that $\overline{v}_F > v^*$ for large enough $\delta$ if and only if $\lambda < \overline{\lambda}$ and $\min\{\underline{c}(\lambda), c^*\} < c < \overline{c}(\lambda)$. Thus, it suffices to show that

$$\{c|\min\{\underline{c}(\lambda),c^*\}<c<\overline{c}(\lambda)\}\;=\;\{c|\underline{c}(\lambda)<c<\overline{c}(\lambda)\}. \tag{33}$$

For $\lambda\le\widetilde{\lambda}$, this follows from $c^*<\underline{c}(\lambda)\Leftrightarrow\overline{c}(\lambda)<c^*$ because

$$\overline{c}\,(\lambda)=h-\ell-\lambda(1-h)\frac{p_G-p_B}{1-p_B}<c^*=\frac{(h-\ell)^2}{1-\ell}$$

is equivalent to

$$\frac{(h-\ell)\,(1-h)}{1-\ell}<\lambda(1-h)\frac{p_G-p_B}{1-p_B}\;\Leftrightarrow\;\frac{(h-\ell)^2}{1-\ell}<\lambda(1-h)\frac{p_G-p_B}{1-p_B}=\underline{c}\,(\lambda)\,.$$

For $\lambda>\widetilde{\lambda}$, we verify (33) by showing $\underline{c}(\lambda)<c^*$ below. First, observe that

$$\underline{c}'(\lambda)\;=\;\frac{h(h-\ell)(h(1-2\lambda)^2-\lambda^2)}{(1-\lambda)^2(\lambda+h-2\lambda h)^2}$$

is positive for $\lambda<\breve{\lambda}$ and negative for $\lambda>\breve{\lambda}$, hence $\underline{c}(\lambda)$ is single-peaked at $\breve{\lambda}=\frac{\sqrt{h}}{1+2\sqrt{h}}\in(0,1/3)$ with a maximum $\underline{c}(\breve{\lambda})=\frac{h(h-\ell)}{(1+\sqrt{h})^2}$.

From $\underline{c}(\widetilde{\lambda})=h(h-\ell)\frac{\sqrt{h(1-h)(1-\ell)\ell}-(1-h)\ell}{(1-h)\sqrt{h(1-h)(1-\ell)\ell}+h^2(1-\ell)}$, we have

$$\frac{c^*-\underline{c}(\widetilde{\lambda})}{h-\ell}=\frac{(h^2+\ell-2h\ell)\big[\sqrt{h(1-h)(1-\ell)\ell}-(1-\ell)h\big]}{(\ell-1)\big[(1-h)\sqrt{h(1-h)(1-\ell)\ell}+h^2(1-\ell)\big]}>0$$

where the inequality follows from $\sqrt{h(1-h)(1-\ell)\ell}<(1-\ell)h$. Hence, $\underline{c}(\lambda)<c^*$ is verified for $\lambda>\widetilde{\lambda}$ if $\breve{\lambda}\le\widetilde{\lambda}$.

Finally, $\breve{\lambda}>\widetilde{\lambda}$ if and only if $\ell<\frac{h^2}{1+2(1-h)\sqrt{h}}$. Moreover, $\frac{c^*-\underline{c}(\breve{\lambda})}{h-\ell}=\frac{h-\ell}{1-\ell}-\frac{h}{(1+\sqrt{h})^2}$ is decreasing in $\ell$ and assumes a positive value of $2\sqrt{h^3}/(1+\sqrt{h})^2$ at $\ell=\frac{h^2}{1+2(1-h)\sqrt{h}}$. Therefore, $\underline{c}(\lambda)<c^*$ obtains for $\lambda>\widetilde{\lambda}$ when $\breve{\lambda}>\widetilde{\lambda}$ as well.

# F Proof of Lemma 2

Observe that

$$\overline{\lambda}=\frac{\sqrt{1+2Y}-1}{(3-\beta)Y}\quad\text{where}\quad Y=\frac{2\,(2h-1)\,(1-\beta)}{(1-h)\,(3-\beta)^2}.$$

Note that $\overline{\lambda}$ decreases in $Y$ and $Y$ increases in $h$, hence $\overline{\lambda}$ decreases in $h$. Also, $\overline{\lambda}\to1/2$ as $\beta\to1$ because $Y|_{\beta=1}=0$ and $\frac{\sqrt{1+2Y}-1}{Y}=\frac{2}{\sqrt{1+2Y}+1}\to1$ as $Y\to0$. In addition,

$$\frac{\partial\overline{\lambda}}{\partial\beta}=\frac{1}{(1-\beta)(3-\beta)^2\sqrt{1+\frac{2Y}{(3-\beta)}}}\left[2\frac{1+2Y-\sqrt{1+2Y}}{Y}-1-\beta\right].$$

Note that the fraction in the bracket is positive and increasing in $Y > -0.5$, and that $Y > Y|_{h=0} = \frac{-2(1-\beta)}{(3-\beta)^2} > -0.5$ since $Y$ increases in $h$. Thus, the expression in the bracket is minimal at $h = 0$ for any given $\beta$, which is calculated as

$$2\frac{1 + 2Y|_{h=0} - \sqrt{1 + 2Y|_{h=0}}}{Y|_{h=0}} - 1 - \beta \;=\; (3-\beta)\frac{\sqrt{5 - 2\beta + \beta^2} - 2}{1-\beta} \;>\; 0$$

where the inequality follows because $5 - 2\beta + \beta^2 > 4$. This proves that $\partial\overline{\lambda}/\partial\beta > 0$.

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